

HIGHER ASSOCIATIVITY OF MOORE SPECTRA

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ABSTRACT. The Moore spectrum $M_p(i)$ is the cofiber of the p^i map on the sphere spectrum. For a fixed p and n , we find a lower bound on i for which a unital A_n -structure on $M_p(i)$ is guaranteed. This bound is dependent on the stable homotopy groups of spheres.

Keywords: Spectra with additional structure

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1. INTRODUCTION

Throughout this paper $(\mathcal{S}p, \wedge, \mathbb{S}^0)$ will denote the the category of \mathbb{S} -modules of [EKMM], which is a modern point-set level closed symmetric monoidal category of spectra. One may also choose to work in other modern point-set categories of spectra, such as the symmetric spectra in simplicial sets of [HSS] or the orthogonal

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Author is supported in part by NSF through grant DMS-1105255.

spectra of [MMSS], however, some of the technical adjustments needed at various stages of the paper may vary depending on the chosen category of spectra.

In $(\mathcal{S}p, \wedge, \mathbb{S}^0)$ a commutative monoid is precisely an E_∞ ring spectrum. The sphere spectrum \mathbb{S}^0 , being the unit, is a perfectly good E_∞ ring spectrum. We are interested in the multiplicative structures of the i -th Moore spectrum at a prime p , $M_p(i)$, the cofiber of the map $p^i : \mathbb{S}^0 \rightarrow \mathbb{S}^0$. The spectrum $M_p(i)$ is analogous to \mathbb{Z}/p^i in abelian groups. Indeed, \mathbb{Z}/p^i is the quotient of the p^i map on the unit \mathbb{Z} in the category $(\mathcal{A}b, \otimes, \mathbb{Z})$. The ring \mathbb{Z}/p^i inherits the commutative and the associative structures from \mathbb{Z} , however $M_p(i)$ does not inherit the E_∞ -structure from \mathbb{S}^0 . A proof can be found in [MNN, Remark 4.3].

Specializing to the case when $i = 1$, it is known that $M_p(1)$ is not even A_∞ (or equivalently E_1). In fact, using Steenrod's squaring operation, one can show that $M_2(1)$ does not even admit a multiplicative structure. Toda [Toda] showed that $M_3(1)$ is not homotopy associative. In general, combining the work of Toda, Kochman [Ko] and Kraines [Kr], one can show that $M_p(1)$ admits an A_{p-1} -structure but not an A_p -structure. An account of the proof using these results can be found in [A1, Example 3.3].

Let us pause here to briefly recall the notion of A_n -structures. Stasheff [STA I, II] describes a hierarchy of coherence for homotopy associative multiplications, called A_n -structures where $1 \leq n \leq \infty$. A unital pairing is A_2 and a homotopy associative pairing is A_3 . Stasheff generalized this sequence by constructing a sequence of spaces \mathcal{K}^n to parameterize higher associativity homotopies. The space \mathcal{K}^n is called the n -th Stasheff polytope (\mathcal{K}^i for $0 \leq i \leq 2$ is a point, \mathcal{K}^3 is the unit interval, \mathcal{K}^4 is a pentagon and so on). An A_n structure on a spectrum X is a sequence of maps

$$\mu_i : \mathcal{K}_+^i \longrightarrow \mathcal{F}(X^{\wedge i}, X)$$

for $0 \leq i \leq n$, with appropriate compatibility criteria. In modern language the Stasheff polytopes and their appropriate subspaces can be put together to form a sequence of unital topological operads \mathcal{A}_n , called the Stasheff A_n -operads. An A_n -structure on a spectrum is an \mathcal{A}_n operad algebra structure on the spectrum.

Hardly anything was known about the A_n -structures of $M_p(i)$ for $i > 1$ before the result in this paper. The only result dates back to 1982 when Oka [O, Theorem 2] proved that $M_2(i)$ admits an A_3 -structure (i.e. a homotopy associative multiplication) for $i \geq 2$. However, experts believe that for every $i > 1$, $M_p(i)$ does not admit any A_∞ -structure. In fact, Mark Mahowald communicated a more general conjecture.

Conjecture 1 (Mahowald). *For any nonzero $\tau \in \pi_{k-1}(\mathbb{S}^0)$ the spectrum $C\tau$, the cofiber of τ , does not admit an A_∞ -structure.*

In this paper, we prove some positive results about the existence of A_n -structures.

Main Theorem 1. *Fix a prime p and an integer $n > 1$. Define the function $o_p(n)$ as*

$$o_p(n) = \#\{k : k \leq 2n - 3, k \text{ odd, and } p\text{-torsion of } \pi_k(\mathbb{S}^0) \text{ is nonzero}\}.$$

When p is odd, $M_p(i)$ admits an \mathcal{A}_n -algebra structure if $i > o_p(n)$. When $p = 2$, $M_2(i + 1)$ admits an \mathcal{A}_n -algebra structure if $i > o_2(n)$.

The following table is a list of values of $o_p(n)$ for $p = 2, 3$ and 5 for small values of n .

| n | $o_2(n)$ | $o_3(n)$ | $o_5(n)$ | n | $o_2(n)$ | $o_3(n)$ | $o_5(n)$ |
|-----|----------|----------|----------|-----|----------|----------|----------|
| 2 | 1 | 0 | 0 | 9 | 6 | 5 | 2 |
| 3 | 2 | 1 | 0 | 10 | 7 | 5 | 2 |
| 4 | 2 | 1 | 0 | 11 | 8 | 6 | 2 |
| 5 | 3 | 2 | 1 | 12 | 9 | 6 | 2 |
| 6 | 4 | 2 | 1 | 13 | 10 | 7 | 3 |
| 7 | 5 | 3 | 1 | 14 | 11 | 7 | 3 |
| 8 | 5 | 4 | 1 | 15 | 12 | 8 | 3 |

TABLE 1. Values of $o_p(n)$ for $p = 2, 3$ and 5 .

We prove the Main Theorem 1 by studying the Moore spectrum $M_p(i)$ as a Thom spectrum. Roughly speaking, a Thom spectrum Mf is always associated to a map $f : X \rightarrow BG$ in $\mathcal{T}op$, where G is a certain kind of topological grouplike object and BG is the associated bar complex. It is the case that Mf admits an A_n -structure if f is an A_n -map, a map that preserves A_n -structure (see Theorem 5.11). We prove that $M_p(i)$ is a Thom spectrum associated to a certain map

$$f_p(i) : S^1 \longrightarrow B\mathcal{G}_p$$

which we will explain in Section 5 (see Lemma 5.8). Using obstruction theory for A_n -maps in $\mathcal{T}op$ developed by Stasheff [STA I, II], we give a lower bound on n for which $f_p(i)$ is an A_n -map.

However, there is no reason to expect all A_n -structures on $M_p(i)$ to be obtained by virtue of being a Thom spectrum. Therefore, we analyse the A_n -structures of $M_p(i)$ through obstruction theory for A_n -structures and A_n -maps in $\mathcal{S}p$. This alternative approach is independent of the Thom spectrum structure of $M_p(i)$. Using this obstruction theory we can prove that (see Theorem 4.18), for an odd prime p and $1 \leq i \leq p$, $M_p(i)$ admits $A_{i(p-1)}$ -structure, and for $i > p$, $M_p(i)$ admits an $A_{p(p-1)}$ -structure. This obstruction theory also allows us to prove that $M_2(i)$ for $i > 1$ admits an A_4 -structure (see Example 3.15). The alternate method fails to give good estimates for A_n -structures for large i for reasons that are explained in Remark 4.16. However, for small values of i it is certainly comparable to the estimates obtained from Main Theorem 1 that involves the Thom spectrum structure of $M_p(i)$. At an odd prime p , both the estimates are same for $1 \leq i \leq p$. At prime 2, the alternate method surprisingly gives a better estimate for A_n -structure on $M_2(3)$ than what we get from Main Theorem 1. We see that $M_2(3)$ as a Thom spectrum admits an A_2 -structure, whereas in general it admits an A_4 -structure. This leads to the question whether all the A_n -structures on $M_p(i)$ can be obtained from its Thom spectrum structure.

The author is not aware of an example of a Thom spectrum which admits A_n -structure, which does not arise from its Thom spectrum structure, but strongly believes that $M_2(2)$ and $M_2(3)$ are potential candidates. However, an example certainly exists if we ask the same question for E_n -structures for Thom spectra, namely the Eilenberg-McLane spectrum $H\mathbb{F}_2$. Indeed, as a Thom spectrum $H\mathbb{F}_2$ admits only an E_2 -structure, as it is a Thom spectrum associated to an E_2 -map $\tau : \Omega^2 S^3 \rightarrow BO$ (see [M]), which is not an E_3 -map. However, $H\mathbb{F}_2$ is an E_∞

ring spectrum. Thus the E_∞ -structure on $H\mathbb{F}_2$ cannot be obtained from its Thom spectrum structure.

The following table lists the highest value of n for which $M_p(i)$ admits A_n -structure for $p = 2, 3$ or 5 and $1 \leq i \leq 7$, that can be concluded from the work in this paper, which includes Main Theorem 1, Theorem 4.18 and Remark 3.15.

| $i \rightarrow$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|-------|-------|----------|----------|----------|----------|----------|
| $p = 2$ | A_1 | A_4 | A_4 | A_4 | A_5 | A_6 | A_8 |
| $p = 3$ | A_2 | A_4 | A_6 | A_7 | A_8 | A_{10} | A_{12} |
| $p = 5$ | A_4 | A_8 | A_{12} | A_{16} | A_{20} | A_{23} | A_{24} |

TABLE 2. A_n -structure that exists on $M_p(i)$ for $p = 2, 3$ and 5 .

Remark. The result in Main Theorem 1 tells us about the existence of A_n -structures on $M_p(i)$ but does not say anything about the non-existence of A_n -structures, which means that Conjecture 1 remains open even for the two-cell complex $M_p(i)$ when $i \geq 2$.

Organization of the paper. In Section 2, we review the Stasheff polytopes, the Stasheff A_n operads and algebras over these operads in a fairly general category. This allows us to develop the notion of A_n -structure simultaneously for objects in $\mathcal{T}op$ and $\mathcal{S}p$.

In Section 3, we describe the obstruction theory for A_n -structures as developed by Stasheff. This obstruction theory gives the best known estimates for A_n -structures on $M_2(2)$ and $M_2(3)$ (see Example 3.15).

Section 4 mainly deals with obstruction theory for A_n -maps. In Subsection 4.1, we describe the obstruction theory for homotopy A_n -maps. In Subsection 4.2, we introduce a method to alter the A_n -structures of two-cell complexes by elements in their homotopy groups. In Subsection 4.3, we make use of all the tools developed in the previous sections to analyse the A_n -structures on $M_p(i)$ resulting in Theorem 4.18. The proof of Theorem 4.18 is independent of the Thom spectrum structure on $M_p(i)$. In Subsection 4.4, we discuss the construction of Stasheff's truncated bar complex which is an important tool to detect A_n -maps in $\mathcal{T}op$. This is needed in the proof of Main Theorem 1.

In Section 5, first we construct the Moore spectrum $M_p(i)$ as a Thom spectrum. Then, we prove the Main Theorem 1 making use of the Thom spectrum structure on $M_p(i)$ as well as the results described in Subsection 4.4.

ACKNOWLEDGMENTS

I am privileged to have discussed this problem with late Mark Mahowald and would like to thank him for many insightful comments regarding this project. I have also benefitted from discussions with Vigliek Angleveit, Tyler Lawson, Akhil Mathew and Haynes Miller. Above all, I would like to thank my adviser Michael Mandell for introducing me to this project as well as helping me out at various stages of this project. This project would have been impossible without his support, encouragement and insight. I would also like to thank Mathematical Science Research Institute for its hospitality during which part of this research was done.

2. STASHEFF A_n OPERADS AND A_n ALGEBRAS

In 1963, Stasheff [STA I, II] introduced a sequence of polytopes \mathcal{K}^n which are now known as Stasheff polytopes. These polytopes are tailor-made to describe a sequence of (non-sigma) operads, which are called the Stasheff A_n -operads.

The Stasheff polytope \mathcal{K}^n , as a topological space, is just homeomorphic to the disk D^{n-2} , but encodes a rich cellular structure which parametrizes a homotopy coherent associative structure. The cells of \mathcal{K}^n are indexed by the set of planar rooted trees with n leaves. The polytopes \mathcal{K}^1 and \mathcal{K}^2 are just one-point spaces. The polytope \mathcal{K}^3 is the unit interval and its cellular structure is described in the picture below.

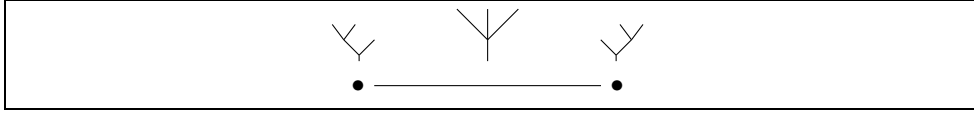


Figure 1: Cellular structure of \mathcal{K}^3 expressed in terms of trees

Let $\mathcal{T}op$ denote the category of compactly generated weakly Hausdorff topological spaces. A product structure on $X \in \mathcal{T}op$ is a map $\mu : X \times X \rightarrow X$. One can also think of the product structure as a map

$$\mu_2 : \mathcal{K}^2(\cong *) \longrightarrow \mathcal{F}(X \times X, X).$$

Thus, the one point set \mathcal{K}^2 parameterizes the multiplication. If this multiplication is homotopy associative then the homotopy can be thought of as a map

$$\mu_3 : \mathcal{K}^3(\cong [0, 1]) \longrightarrow \mathcal{F}(X \times X \times X, X)$$

whose evaluation at the end points are given by the formulas $\mu_3(0) = \mu_2 \circ (\mu_2 \times 1)$ and $\mu_3(1) = \mu_2 \circ (\mu_2 \times 1)$.

The polytope \mathcal{K}^4 is the pentagon. Given a multiplication μ_2 , there are five different four-fold multiplications, producing five different maps from $X^{\times 4}$ to X . These multiplications can be encoded by five different binary trees with four leaves. These trees label the five vertices of \mathcal{K}^4 (see Figure 4.1). Moreover, if the multiplication is homotopy associative, i.e. μ_3 exists, then one can construct homotopies between any two four-fold multiplications. These homotopies can be glued together to give a map

$$\partial\mu_4 : S^1 \cong \partial\mathcal{K}^4 \longrightarrow \mathcal{F}(X^{\times 4}, X).$$

The edges of \mathcal{K}^4 are denoted by the planar rooted trees with 4 leaves and one internal vertex. One should note that the arrangement is such that the tree that represents the edge can be obtained by collapsing one of the edges of the trees that represent the adjacent vertices. If the map $\partial\mu_4$ is homotopic to a constant map, then we can use this homotopy to obtain a map

$$\mu_4 : \mathcal{K}^4 \longrightarrow \mathcal{F}(X^{\times 4}, X).$$

Thus \mathcal{K}^4 parameterizes homotopy coherence of the associativity among the four-fold multiplications.

In general the cells of \mathcal{K}^n are in one-to-one correspondence with the planar rooted trees with n -leaves. More specifically, the codimension k cells are in bijection with the planar rooted trees with n leaves and k internal vertices.

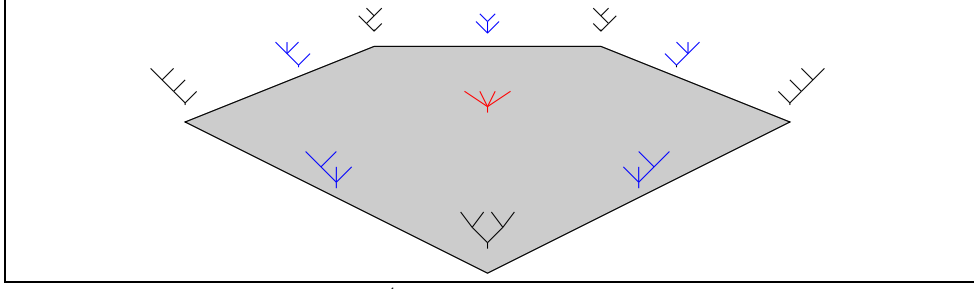


Figure 2: \mathcal{K}^4 with its cells indexed by trees

Notation 2.1. Let T_k be the collection of planar rooted trees with k leaves and

$$T_* = \bigcup_k T_k.$$

For each $t \in T_k$, let $\mathcal{K}(t)$ denote the corresponding cell of \mathcal{K}^k .

Definition 2.2. A *corolla* is a planar rooted tree with no internal vertices.

There is exactly one corolla in T_k for all $k \in \mathbb{N}$. If $t \in T_k$ is a corolla then $\mathcal{K}(t)$ is the Stasheff polytope \mathcal{K}^k . For any other tree $t \in T_k$, we can obtain a set of corollas by breaking the tree off at each vertex. We call this set the *corolla decomposition* of t and denote it by $C(t)$.

Example 2.3. If t is the tree



then $C(t) = \{\searrow, \swarrow, \vee, \wedge\}$.

The cell $\mathcal{K}(t) \subset \mathcal{K}^n$ is homeomorphic to

$$\prod_{s \in C(t)} \mathcal{K}(s) = \prod_{s \in C(t)} \mathcal{K}^{l(s)}$$

where $l(s)$ denotes the number of leaves in the corolla s . This product is unique up to association. There are various models for Stasheff polytope \mathcal{K}^n . The first one is of course due to Stasheff [STA I, II]. Other prominent models include [BV, CFZ, Lod, Tonk].

2.1. Non- Σ Operads. The Stasheff polytopes can be used to parameterize homotopy coherence of the higher associativities, which can be best explained with the language of operads. Operads often come with symmetries, and those that do not are called non- Σ operads. Since we mostly work with non- Σ operads, we will save the term ‘operad’ to refer to non- Σ operads.

Let Δ denote the category of the finite, non-empty, totally ordered sets with order preserving maps as morphisms, and let Δ_+ be the category of finite totally ordered sets, i.e., the empty set is included in Δ_+ . Let $iso(\Delta_+)$ denote the subcategory of Δ_+ whose objects are objects of Δ_+ but morphisms are just the isomorphisms of Δ_+ .

Definition 2.4. A *sequence* in $\mathcal{T}op$ is a functor

$$\mathcal{O} : iso(\Delta_+) \longrightarrow \mathcal{T}op$$

and the n -th term of the sequence $\mathcal{O}(n)$ is the image of the isomorphism class of $\{0 < 1 < \dots < n-1\}$. A map $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ of sequences is simply a collection of maps $f_i : \mathcal{O}_1(n) \rightarrow \mathcal{O}_2(n)$ in $\mathcal{T}op$.

Notation 2.5. Given a sequence $\mathcal{O} : iso(\Delta_+) \rightarrow \mathcal{T}op$ and a map $f : S \rightarrow T$ where S and T are objects in Δ_+ , define

$$\mathcal{O}[f] = \prod_{t \in T} \mathcal{O}(f^{-1}(t)).$$

Definition 2.6. Given two sequences \mathcal{O}_1 and \mathcal{O}_2 in $\mathcal{T}op$, their composition product $\mathcal{O}_1 \circ \mathcal{O}_2$ is given by the formula

$$(\mathcal{O}_1 \circ \mathcal{O}_2)(S) = \coprod_{f: S \rightarrow T} \mathcal{O}_1(T) \times \mathcal{O}_2[f]$$

where S is a finite totally ordered set and the coproduct runs over all isomorphism classes of a totally ordered set T and order preserving maps $f : S \rightarrow T$.

The composition product \circ is monoidal as the coproduct \sqcup in $\mathcal{T}op$ distributes over the monoidal product \times . Let \mathcal{I} be the sequence with $\mathcal{I}(1) = *$ and $\mathcal{I}(n) = \emptyset$ for $n \neq 1$. The sequence \mathcal{I} has the special property that

$$\mathcal{I} \circ \mathcal{O} \cong \mathcal{O} \cong \mathcal{O} \circ \mathcal{I}.$$

Definition 2.7. An *nonunital operad* is a sequence \mathcal{O} with $\mathcal{O}(0) = \emptyset$ and a multiplication map

$$\mathcal{O} \circ \mathcal{O} \longrightarrow \mathcal{O}$$

which is associative.

Definition 2.8. A *unital operad* is a sequence \mathcal{O} with $\mathcal{O}(0) = *$ that admits a unit map $\mathcal{I} \rightarrow \mathcal{O}$ and a multiplication map

$$\mathcal{O} \circ \mathcal{O} \longrightarrow \mathcal{O}$$

which is associative and compatible with the unit map.

For any $S \in \Delta_+$ and $s \in S$, define

$$S \cup_s T = S - \{s\} \sqcup T$$

with the following ordering. The ordering within the set $S - \{s\}$ and T is preserved. For $s' \in S - \{s\}$ and $t \in T$, $s' < t$ if $s' < s$ in S and $s' > t$ if $s' > s$. Let

$$f : S \cup_s T \longrightarrow S$$

be the order preserving map which sends every element of T to s and is the identity elsewhere. An operad structure on a sequence \mathcal{O} , by definition, determines a map

$$\circ_s : \mathcal{O}(S) \times \mathcal{O}(T) \longrightarrow \mathcal{O}(S \cup_s T).$$

In particular when $S \cong \{0 < \dots < n-1\}$, $T \cong \{0 < \dots < k-1\}$ and $s = i-1$, we will denote \circ_s by \circ_i (as $s = i-1$ is the i -th object with the ordering in S). It is shown (see [MSS, §1.7.1]) that the operations $\{\circ_i : i \geq 1\}$ determine and are determined by the operad structure on \mathcal{O} .

Remark 2.9. Some readers might be familiar with the following alternative definition of operad. An *operad* \mathcal{O} is a sequence of spaces $\mathcal{O}(n)$ for $n \geq 0$ together with the data of continuous functions

$$\gamma(n; j_1, \dots, j_n) : \mathcal{O}(n) \times (\mathcal{O}(j_1) \times \dots \times \mathcal{O}(j_n)) \longrightarrow \mathcal{O}(j)$$

where $j = j_1 + \dots + j_n$, which satisfy

$$\gamma(\gamma(n; j_1, \dots, j_n); i_1, \dots, i_j) = \gamma(n; \gamma(j_1; i_1, \dots, i_{j_1}), \dots, \gamma(j_n; i_{j-j_n+1}, \dots, i_j))$$

and a specified identity element $1 \in \mathcal{O}(1)$ which satisfies,

$$\gamma(1; n)(1, x) = x \text{ and } \gamma(n; 1, \dots, 1)(x, 1, \dots, 1) = x.$$

Under this definition the map \circ_i is the map

$$\gamma(n, 1, \dots, 1, k, 1, \dots, 1) : \mathcal{O}(n) \times (\mathcal{O}(1)^{\times i-1} \times \mathcal{O}(k) \times \mathcal{O}(1)^{\times n-i}) \longrightarrow \mathcal{O}(n+k-1).$$

Example 2.10 (Endomorphism operad). For every object $X \in \mathcal{T}op$, the *endomorphism operad* $\mathcal{E}(X)$ has the n -th space as

$$\mathcal{E}(X)(n) = Hom_{\mathcal{T}op}(X^{\times n}, X)$$

and the map

$$\circ_i : \mathcal{E}(X)(n) \times \mathcal{E}(X)(m) \longrightarrow \mathcal{E}(X)(n+m-1)$$

sends (f, g) to the composite

$$X^{\times(n+m-1)} \xrightarrow{\overbrace{1 \times \dots \times 1}^{i-1} \times g \times \overbrace{1 \times \dots \times 1}^{n-i}} X^{\times n} \xrightarrow{f} X.$$

Definition 2.11. A morphism between two operads \mathcal{O} and \mathcal{P} is a map of sequences $f : \mathcal{O} \rightarrow \mathcal{P}$ such that the diagram

$$\begin{array}{ccc} \mathcal{O} \circ \mathcal{O} & \xrightarrow{f \circ f} & \mathcal{P} \circ \mathcal{P} \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P} \end{array}$$

commutes.

2.2. Stasheff A_n operads. The non-unital Stasheff A_∞ operad, denoted by \mathcal{A}_∞° , is the sequence

$$\mathcal{A}_\infty^\circ(k) = \begin{cases} \emptyset & \text{if } k = 0 \\ \mathcal{K}^k & \text{if } k \neq 0. \end{cases}$$

In order to describe the operad structure, we will define

$$\circ_i : \mathcal{K}^n \times \mathcal{K}^k \longrightarrow \mathcal{K}^{n+k-1}$$

for $n \geq 0, k \geq 0$ and $1 \leq i \leq n$ as pointed out in Remark 2.23. Any cell in $\mathcal{K}^n \times \mathcal{K}^k$ can be indexed by a tuple (t_1, t_2) of planar rooted trees where t_1 has n leaves and t_2 has k leaves. The map \circ_i will map this cell homeomorphically onto a cell of \mathcal{K}^{n+k-1} which is indexed by the tree obtained by concatenating t_2 at the i -th leaf of t_1 . Thus we have described the map \circ_i at the cellular level.

A point-set level description of \circ_i is established in [STA I, II] as well as in [BV]. We give another description using those models of Stasheff polytopes, where they are realized as the convex hulls on their set of vertices embedded in \mathbb{R}^{n-2} (e.g. [Lod]). Define the *special point* $p(C)$, for any cell C of the Stasheff polytope \mathcal{K}^n , to

be the vertex of C represented by the binary tree which is skewed the most towards the left.

Example 2.12. If C is the cell of \mathcal{K}^5 indexed by the tree



then the special point of C is the 0-cell of C and indexed by the binary tree



For a cell $C = C_1 \times \cdots \times C_n$ in the product $\mathcal{K}^{i_1} \times \cdots \times \mathcal{K}^{i_n}$, define the special point C to be the product of the special points of C_i i.e.

$$p(C) = (p(C_1), \dots, p(C_n)).$$

Note that the map

$$(2.13) \quad \circ_i : \mathcal{K}^n \times \mathcal{K}^k \longrightarrow \mathcal{K}^{n+k-1}$$

sends the special point of a cell $C_1 \times C_2$ of $\mathcal{K}^n \times \mathcal{K}^k$ to the special point of $\circ_i(C_1 \times C_2)$, i.e.

$$p(\circ_i(C_1 \times C_2)) = \circ_i(p(C_1 \times C_2)).$$

We will use this property to define the point-set level map \circ_i by induction on the dimension of cells. The cellular description of the map \circ_i already determines the map on the 0-cells. Inductively, assume that the map on $(j-1)$ -cells is defined. Then we extend \circ_i to any j -cell $C \subset \mathcal{K}^n \times \mathcal{K}^k$ as follows. Since C is the convex hull on the set of its vertices, any point $x \in C \setminus \partial C$ can be expressed uniquely as a linear combination $tp(C) + (1-t)y$, where $y \in \partial C$. Since $y \in \partial C$, y is in some $(j-1)$ -cell and hence by inductive step $\circ_i(y)$ is already determined. Now define,

$$(2.14) \quad \circ_i(x) = t \circ_i(p(C)) + (1-t) \circ_i(y).$$

It is easy to check that the point-set level maps defined inductively satisfy the usual compatibility conditions that are required to establish \mathcal{A}_∞° into an operad.

Notation 2.15. Let $T_k\langle n \rangle \subset T_k$ consists of trees which have at most n decendants (including leaves) from each vertex and

$$T_*\langle n \rangle = \bigcup_k T_k\langle n \rangle.$$

In other words, $T_*\langle n \rangle$ consists of those trees whose corolla decomposition consists of corollas with at most n leaves.

For $1 \leq n < \infty$, define the *non-unital Stasheff A_n -operad* \mathcal{A}_n° as the sequence

$$\mathcal{A}_n^\circ(k) = \begin{cases} \emptyset & \text{if } k = 0 \\ \mathcal{K}^k\langle n \rangle & \text{if } k \neq 0 \end{cases}$$

where

$$\mathcal{K}^k\langle n \rangle = \bigcup_{t \in T_k\langle n \rangle} \mathcal{K}(t) \subset \mathcal{K}^k.$$

Remark 2.16. When $k \leq n$, the set of trees with at most n descendants from each vertex, include all the trees with k leaves, i.e. $T_k = T_k\langle n \rangle$. Therefore,

$$\mathcal{K}^k\langle n \rangle = \mathcal{K}^k$$

for $k \leq n$.

Remark 2.17. When $k = n + 1$, all trees with k leaves except the one with no internal vertices belong to $T_{n+1}\langle n \rangle$. Hence,

$$\mathcal{K}^{n+1}\langle n \rangle = \partial\mathcal{K}^{n+1}.$$

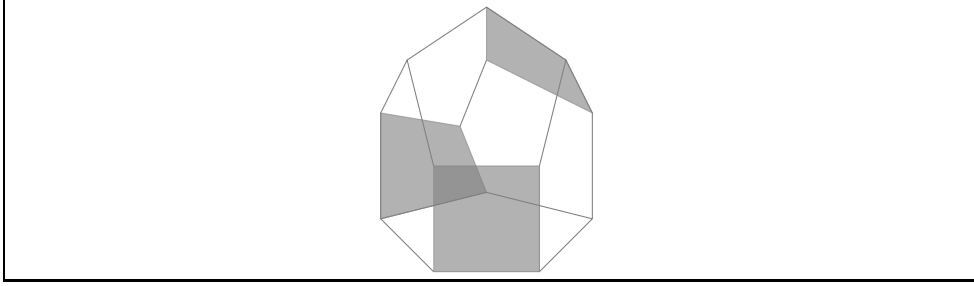


Figure 3: The complex $\mathcal{K}^5\langle 3 \rangle$

Observe that for a pair (t_1, t_2) in $T_*\langle n \rangle$, concatenating t_2 at any leaf of t_1 results in a tree which belongs to $T_*\langle n \rangle$. In other words, $T_*\langle n \rangle$ is closed under the map \circ_i . It follows that the \circ_i restricted to $\mathcal{K}^k\langle n \rangle$ determine the structure maps of the nonunital A_n -operad \mathcal{A}_n° .

The nonunital operad \mathcal{A}_n° can be extended to a unital operad for any $1 \leq n \leq \infty$. The *unital A_n -operad* \mathcal{A}_n can be thought of as a sequence

$$\mathcal{A}_n(k) = \mathcal{K}^k\langle n \rangle$$

for $n \geq 0$ with the convention that $\mathcal{K}^0\langle n \rangle = *$. To describe \mathcal{A}_n as an extension of \mathcal{A}_n° , we need to define additional set of coherent maps

$$s_i := \circ_i : \mathcal{K}^k\langle n \rangle \times \mathcal{K}^0\langle n \rangle \longrightarrow \mathcal{K}^{k-1}\langle n \rangle$$

for each $k \geq 1$ and $1 \leq i \leq k$, called the i -th degeneracy map. The i -th degeneracy map at the cellular level, is simply a map between the collection of trees

$$T_k\langle n \rangle \longrightarrow T_{k-1}\langle n \rangle$$

obtained by ‘deleting the i -th leaf’ of t . In [STA I, II] as well as [BV], the pointset level degeneracy maps for their respective models of non-unital A_n -operad have been defined. For our model, we need to define the degeneracy maps which are compatible with the maps of Equation 2.14.

Notice that the special point of any cell of $\mathcal{K}^k\langle n \rangle$ maps to the special point of the target cell of $\mathcal{K}^k\langle n-1 \rangle$. Using the fact the Stasheff polytope \mathcal{K}^k that we work with is the convex hull on the set of its vertices embedded in \mathbb{R}^{k-2} , we can induct on the dimension of the cells to define s_i , just like we defined \circ_i . The cellular level description of the map s_i determines the map on the 0-cells. Inductively, assume that the map s_i is determined on $(d-1)$ -cells. For a d -cell C of $\mathcal{K}^k\langle n \rangle$, we can uniquely express any point $x \in C \setminus \partial C$ as

$$x = tp(C) + (1-t)y.$$

Since $y \in \partial C$, y belongs to some $(d-1)$ -cell and hence $s_i(y)$ is determined by the inductive hypothesis. Now define

$$s_i(x) = tp(s_i(C)) + (1-t)s_i(y).$$

One can check that the maps s_i are compatible with the maps of Equation 2.13 in a way that makes \mathcal{A}_n a unital operad.

2.3. An abstract category. For the purpose of this paper we need to make sense of A_n -structures on objects in $\mathcal{T}op$ and $\mathcal{S}p$. In order to do so simultaneously, we choose to work in an arbitrary category which has the bare minimum properties necessary to define A_n -structures.

Let $(\mathcal{C}, \mathbf{I}, \otimes)$ be a closed symmetric monoidal category which satisfies the following conditions,

- (C1) there exists an initial object $\star \in \mathcal{C}$, such that $\star \otimes P \cong \star \cong P \otimes \star$ for any object $P \in \mathcal{C}$,
- (C2) the category \mathcal{C} is closed under finite limits and colimits,

with the additional structure of

- (S1) a monoidal functor

$$F : \mathcal{T}op \longrightarrow \mathcal{C}$$

with $F(\emptyset) = \star$, which admits a right adjoint

$$G : \mathcal{C} \longrightarrow \mathcal{T}op.$$

The category $(\mathcal{T}op, \times, *)$ is a trivial example of such a category, where the functors F and G are identity functors. The category $(\mathcal{T}op_*, \wedge, S^0)$ is another such example, where the functor F and G are ‘adding a disjoint basepoint’ functor and ‘forgetting the basepoint’ functor respectively. The category $(\mathcal{S}p, \wedge, S^0)$ is also an example. The functor

$$S[_] : \mathcal{T}op \longrightarrow \mathcal{S}p,$$

where $S[X] = \Sigma_+^\infty X$ should play the role of F , which has a right adjoint.

Warning 2.18. The right adjoint is not the zeroth space functor. However, if the category $(\mathcal{T}op, *, \times)$ is replaced with the category of $*$ -modules $(\mathcal{M}, *, \times_{\mathcal{L}})$, which is Quillen equivalent to $\mathcal{T}op$, then the right adjoint to $*$ -modules is naturally weakly equivalent to the zeroth space functor. A detailed discussion can be found in [Lind] (also see [ABGHR, Section 3]).

For convenience we introduce the notion of *external product*, which is a bifunctor

$$\boxtimes : \mathcal{T}op \times \mathcal{C} \longrightarrow \mathcal{C}$$

where $X \boxtimes P = F(X) \otimes P$. Let

$$hom_{\mathcal{C}}(,) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$

be the internal homomorphism functor of \mathcal{C} . The functor G allows us to define function spaces for any two objects $P, Q \in \mathcal{C}$ by setting

$$\mathcal{F}(P, Q) = G(hom_{\mathcal{C}}(P, Q)).$$

For $X \in \mathcal{T}op$ and $Q \in \mathcal{C}$ we define

$$\mathcal{F}(X, Q) = hom_{\mathcal{T}op}(X, G(Q)).$$

Now assume that \mathcal{C} satisfies an additional condition,

- (C3) the object \star is the zero object (both the initial and the final object) in \mathcal{C} .

Then for any two objects P and Q we have a map

$$P \longrightarrow \star \longrightarrow Q$$

which is called the *zero map*. The map $i : \star \rightarrow \mathcal{F}(P, Q)$ adjoint to the zero map under the adjunction isomorphism

$$\text{hom}_{\mathcal{C}}(\mathbf{I} \otimes P, Q) \cong \text{hom}_{\mathcal{T}op}(\star, \mathcal{F}(P, Q))$$

serves as a natural choice for the basepoint of the function space $\mathcal{F}(P, Q)$.

In \mathcal{C} we define a homotopy between two maps $f, g : P \rightarrow Q$ as a map

$$H : [0, 1]_+ \longrightarrow \mathcal{F}(P, Q)$$

such that $H(0) = \hat{f}$ and $H(1) = \hat{g}$, where \hat{f} and \hat{g} are adjoint to f and g respectively. Define the homotopy class of maps from P to Q to be the set

$$[P, Q] = \mathcal{F}(P, Q) / \simeq$$

where $f \simeq g$ if there exists a homotopy between f and g . Thus, we can define $h\mathcal{C}$, the *homotopy category of \mathcal{C}* , as the category whose objects are the objects of \mathcal{C} and morphisms between two objects are the homotopy class of maps

$$\text{Mor}_{h\mathcal{C}}(P, Q) := [P, Q].$$

A map $f : P \rightarrow Q$ is a *cofibration* if f satisfies the homotopy extension property (**HEP**), i.e. the diagram

$$\begin{array}{ccc} P & \xrightarrow{i_0} & [0, 1] \boxtimes P \\ \downarrow f & \nearrow \alpha & \downarrow \\ Q & \xrightarrow{i_0} & [0, 1] \boxtimes Q \end{array} \quad \begin{array}{c} \nearrow H \\ \nwarrow \tilde{H} \end{array}$$

R

has a solution. For an arbitrary map $f : P \rightarrow Q$ we define the *cofiber* as

$$Cf := P \cup_{\{0\} \boxtimes f} ([0, 1] \boxtimes Q) / (1 \boxtimes Q).$$

A *cofiber sequence* consists of a pair of composable maps

$$P \xrightarrow{f} Q \xrightarrow{g} R$$

in \mathcal{C} such that we have a homotopy equivalence of R with Cf under Q . It is straightforward to verify that if $P \rightarrow Q \rightarrow R$ is a cofiber sequence then

$$[R, Z] \xrightarrow{g^*} [Q, Z] \xrightarrow{f^*} [P, Z]$$

is an exact sequence of sets.

2.4. A_n algebras. Let $(\mathcal{C}, \mathbf{I}, \otimes)$ be a closed symmetric monoidal category which satisfies **(C1)**, **(C2)**, **(C3)** and **(S1)** as defined in Section 2.3. For any $P \in \text{Obj}(\mathcal{C})$, we can associate a topological operad, called the endomorphism operad $\mathcal{E}(P)$ associated to P , whose n -th space is

$$\mathcal{E}(P)(n) = \mathcal{F}(P^{\otimes n}, P)$$

and structure maps are as usual.

Definition 2.19. Let \mathcal{O} be a topological operad. An \mathcal{O} -algebra structure on $P \in \text{Obj}(\mathcal{C})$ is a map of operads

$$f : \mathcal{O} \longrightarrow \mathcal{E}(P).$$

Remark 2.20. Equivalently, one can define an \mathcal{O} -algebra structure on $P \in \text{Obj}(\mathcal{C})$ by producing maps

$$f_n : \mathcal{O}(n) \boxtimes P^{\otimes n} \longrightarrow P$$

which satisfy the usual compatibility criterias. The map f_n is adjoint to the map $f(n) : \mathcal{O}(n) \rightarrow \mathcal{E}(P)(n)$ of Definition 2.19, under the adjunction isomorphism

$$\text{hom}_{\mathcal{T}op}(\mathcal{O}(n), G(\text{hom}_{\mathcal{C}}(P^{\otimes n}, P))) \cong \text{hom}_{\mathcal{C}}(\mathcal{O}(n) \boxtimes P^{\otimes n}, P).$$

Definition 2.21. An object P is said to admit a unital A_n -structure (resp. non-unital A_n -structure) if P admits an \mathcal{A}_n -algebra (resp. \mathcal{A}_n° -algebra) structure on P .

Notation 2.22. Let the structure maps for an \mathcal{A}_n -algebra structure or \mathcal{A}_n° -algebra structure be denoted by

$$\mu_i \langle n \rangle : \mathcal{K}^i \langle n \rangle \boxtimes X^{\otimes i} \longrightarrow X$$

where $1 \leq n \leq \infty$. We will abusively denote $\mu_i \langle n \rangle$ by μ_i when $i \leq n$, as $\mathcal{K}^i \langle n \rangle = \mathcal{K}^i$ for $i \leq n$. By further abuse of notations, we use the same notation $\mu_i \langle n \rangle$ (or μ_i), to denote the adjoint map

$$\widehat{\mu_i \langle n \rangle} : \mathcal{K}^i \langle n \rangle_+ \longrightarrow \mathcal{F}(P^{\otimes i}, P).$$

Remark 2.23. In order to define a unital A_n -structure on P , it is enough to define

$$\mu_i : \mathcal{K}_+^i = \mathcal{A}_n(i)_+ \longrightarrow \mathcal{F}(P^{\otimes i}, P)$$

for $0 \leq i \leq n$, which satisfy the usual compatibility criteria. Since, any cell in $\mathcal{K}^i \langle n \rangle$ for $i > n$, is homeomorphic to the product of Stasheff polytopes \mathcal{K}^i for $i \leq n$, which is unique up to association, the map $\mu_i \langle n \rangle$ for $i > n$, is determined by the maps μ_1, \dots, μ_n .

Remark 2.24 (\mathcal{A}_n -algebras in $\mathcal{T}op$). Note that the condition **(C3)** is not satisfied by $\mathcal{T}op$ since the initial object \emptyset is not the final object. In fact the final object is the one-point set $*$. We can still make sense of \mathcal{A}_n -algebras in $\mathcal{T}op$ with our settings, simply by viewing $X \in \mathcal{T}op$ as an object of $\mathcal{T}op_*$ by adding a disjoint basepoint. In other words, an \mathcal{A}_n -algebra structure on X in $(\mathcal{T}op, *, \times)$ is simply an \mathcal{A}_n -algebra structure on X_+ in $(\mathcal{T}op_*, S^0, \wedge)$.

3. OBSTRUCTION THEORY FOR A_n -STRUCTURES

Throughout this section we work in a closed symmetric monoidal category $(\mathcal{C}, \mathbf{I}, \otimes)$ which satisfies the conditions **(C1)**, **(C2)**, **(C3)** and **(S1)** as described in the Section 2.3. Suppose $X \in \mathcal{C}$ admits an A_{n-1} -structure (unital or nonunital), we want to know, under what conditions this A_{n-1} -structure extends to an A_n -structure.

3.1. Nonunital Obstruction Theory. A nonunital A_{n-1} -structure determines a map

$$\partial\mu_n = \mu_{n-1}\langle n \rangle : \partial\mathcal{K}^n \boxtimes X^{\otimes n} \longrightarrow X$$

as $\mathcal{A}_{n-1}^\circ(n) = \mathcal{K}^n\langle n-1 \rangle = \partial\mathcal{K}^n$. If the map $\partial\mu_n$ extends to a map μ_n as in the diagram

$$\begin{array}{ccc} \partial\mathcal{K}_+^n & \xrightarrow{\partial\mu_n} & \mathcal{F}(X^{\otimes n}, X) \\ \downarrow & \nearrow \mu_n & \\ \mathcal{K}_+^n & & \end{array}$$

or equivalently $\partial\mu_n$ is homotopic to a constant, then by Remark 2.23 we have extended the A_{n-1} -structure to a nonunital A_n -structure. This observation can be summarized as the following theorem.

Theorem 3.1 (Stasheff). *An \mathcal{A}_{n-1}° -algebra structure on $X \in \text{Obj}(\mathcal{C})$ extends to \mathcal{A}_n° -algebra structure if and only if the map*

$$\mu_n\langle n-1 \rangle : \partial\mathcal{K}_+^n \longrightarrow \mathcal{F}(X^{\otimes n}, X)$$

is homotopic to a constant. In other words, the obstruction to extending \mathcal{A}_{n-1}° -algebra structure to \mathcal{A}_n° -algebra structure is the homotopy class $[\mu_n\langle n-1 \rangle]$.

3.2. Unital obstruction theory. The unital obstruction theory for A_n -structures is more involved compared to the nonunital case. We first illustrate this fact in the following remark, by specializing ourselves to the case when $X \in \mathcal{T}op$.

Remark 3.2. Suppose that $X \in \mathcal{T}op$ admits a unital \mathcal{A}_{n-1} -structure. Let $\mu_0 : * \rightarrow X$ be the unit map and let $u = \mu_0(*)$ be the unit. Notice that the diagram

$$\begin{array}{ccc} \mathcal{K}^n\langle n-1 \rangle = \partial\mathcal{K}^n & \xrightarrow{\mu_n\langle n-1 \rangle} & F(X^{\times n}, X) \\ \partial s_i \downarrow & & \downarrow \delta_i \\ \mathcal{K}^{n-1} & \xrightarrow{\mu_{n-1}} & F(X^{\times n-1}, X) \end{array}$$

commutes, where $s_i : \mathcal{K}^n \rightarrow \mathcal{K}^{n-1}$ is the i -th degeneracy map and δ_i is the map induced by $1^{\times i-1} \times \mu_0 \times 1^{\times n-i}$. Hence the adjoint of the composite map $\delta_i \circ \mu_n\langle n-1 \rangle$, which is the composite

$$\partial\mathcal{K}^n \times X^{\times(n-1)} \xrightarrow{1^{\times i-1} \times \mu_0 \times 1^{\times n-i}} \partial\mathcal{K}^n \times X^n \xrightarrow{\mu_n\langle n-1 \rangle} X,$$

extends to a map

$$\mu_{n,i} : \mathcal{K}^n \times X^{\times n-1} \longrightarrow X$$

for $1 \leq i \leq n$, as $\delta_i \circ \mu_n\langle n-1 \rangle$ factors through the contractible space \mathcal{K}^{n-1} . In fact, the map adjoint to

$$\begin{aligned} \mu_{n-1} \circ s_i : \mathcal{K}^n &\longrightarrow \mathcal{K}^{n-1} \longrightarrow F(X^{\times(n-1)}, X) \\ \mu_n^{[n-1]} : \mathcal{K}^n &\times X_{[n-1]}^{\times n} \longrightarrow X \end{aligned}$$

where $X_{[n-1]}^{\times n} = \{(x_1, \dots, x_n) : x_i = u \text{ for some } i\} \subset X^{\times n}$. Therefore, in order to extend the unital A_{n-1} -structure to a unital A_n -structure, we need to produce a map

$$\mu_n : \mathcal{K}^n \times X^{\times n} \longrightarrow X$$

with the additional constraint that when restricted to $\mathcal{K}^n \times X_{[n-1]}^{\times n}$ it is the map $\mu_n^{[n-1]}$. This additional criterion is the striking difference between the obstruction theories for unital A_n -structure and nonunital A_n -structure.

We will now set up this theory for \mathcal{C} . For a based object $X \in \mathcal{C}$, we construct the object $X_{[i]}^{\otimes n}$ as follows. Let $[n]$ be the category whose objects are ordered subsets of the ordered set $\{1 < \dots < n\}$ and morphisms are the inclusion maps. Let $C(n, i)$ be the full subcategory of $[n]$ whose objects are sets with at most i elements.

Definition 3.3. Let $X \in \mathcal{C}$ admit a unit map $\mu_0 : I \rightarrow X$ which is a cofibration. Then one can define a functor

$$F_{n,i}^X : C(n, i) \longrightarrow \mathcal{C}$$

where $F_{n,i}^X(\{i_1 < \dots < i_k\}) = X_1 \otimes \dots \otimes X_n$ where $X_j = X$ if $j \in \{i_1 < \dots < i_k\}$ and $X_j = I$ otherwise. Define

$$X_{[k]}^{\otimes n} = \operatorname{colim}_{C(n,k)} F_{n,k}^X.$$

The above definition makes sense as the map \mathcal{C} is closed under finite colimits (Condition **(C2)**). It is necessary for μ_0 to be a cofibration to make sure that the natural map

$$X_{[k]}^{\otimes n} \longrightarrow X_{[n]}^{\otimes n} \cong X^{\otimes n}$$

is a cofibration.

Lemma 3.4. *If X admits an \mathcal{A}_{n-1} -algebra structure (i.e. a unital A_{n-1} -structure) then the \mathcal{A}_{n-1} -algebra structure determines a map*

$$\mu_n^{[n-1]} : \mathcal{K}^n \boxtimes X_{[n-1]}^{\otimes n} \longrightarrow X.$$

Proof. Let $T \in \operatorname{Obj}(C(n, n-1))$. Corresponding to the morphism

$$T \hookrightarrow \{1 < \dots < n\},$$

there is a degeneracy map of Stasheff polytopes

$$s_T : \mathcal{K}^n \longrightarrow \mathcal{K}^k$$

which is defined as follows. If $\{1 < \dots < n\} \setminus T = \{j_1 < \dots < j_{n-k}\}$ then s_T corresponds to the composite

$$\mathcal{K}^n \xrightarrow{s_{j_{n-k}}} \dots \xrightarrow{s_{j_1}} \mathcal{K}^k.$$

Corresponding to $T \hookrightarrow \{1 < \dots < n\}$ we also have the map

$$F_{n,n-1}^X(T \hookrightarrow \{1 < \dots < n\}) : F_{n,n-1}^X(T) \cong X^{\otimes k} \longrightarrow F_{n,n-1}^X([n]) \cong X^{\otimes n}.$$

which induces the map

$$\delta_T : \mathcal{F}(X^{\otimes n}, X) \longrightarrow \mathcal{F}(X^{\otimes k}, X).$$

The \mathcal{A}_{n-1} -algebra structure guarantees that the diagram

$$\begin{array}{ccc} \partial \mathcal{K}_+^n & \xrightarrow{\mu_n \langle n-1 \rangle} & \mathcal{F}(X^{\otimes n}, X) \\ \partial s_T \downarrow & & \downarrow \delta_T \\ \mathcal{K}_+^k & \xrightarrow{\mu_k} & \mathcal{F}(X^{\otimes k}, X) \end{array}$$

$$\mu_n : \mathcal{K}^n \boxtimes X^{\otimes n} \longrightarrow X$$

such that the diagram

$$\begin{array}{ccc} \phi_n(X) & \xrightarrow{\eta} & \mathcal{K}^n \boxtimes X^{\otimes n} \\ & \searrow \phi(\mu_n) & \downarrow \mu_n \\ & & X \end{array}$$

commutes. Suppose there is an object $\sigma_n(X)$ such that

$$(3.7) \quad \sigma_n(X) \xrightarrow{\iota} \phi_n(X) \longrightarrow \mathcal{K}^n \boxtimes X^{\otimes n}$$

is a cofiber sequence. Then, from the diagram

$$\begin{array}{ccccc} \sigma_n(X) & \xrightarrow{\iota} & \phi_n(X) & \xrightarrow{\eta} & \mathcal{K}^n \boxtimes X^{\otimes n} \\ & & \downarrow \phi(\mu_n) & \nearrow \mu_n & \\ & & X, & & \end{array}$$

it is clear that the \mathcal{A}_{n-1} -algebra structure on X extends to \mathcal{A}_n -algebra structure if the composite $\phi(\mu_n) \circ \iota$ is null homotopic. Thus we have established the following theorem.

Theorem 3.8 (Stasheff). *Let $X \in \text{Obj}(\mathcal{C})$ be an \mathcal{A}_{n-1} -algebra such that $\sigma_n(X)$ (as defined in Equation 3.7) exists. Let $\phi_n(X)$, $\phi(\mu_n)$ and ι are as defined in Equation 3.5, Equation 3.6 and Equation 3.7 respectively. The \mathcal{A}_{n-1} -algebra structure on X extends to an \mathcal{A}_n -algebra structure if and only if the map*

$$\sigma_n(X) \xrightarrow{\iota} \phi_n(X) \xrightarrow{\phi(\mu_n)} X,$$

is homotopic to the zero map. In other words the obstruction to extending an \mathcal{A}_{n-1} -algebra structure on X to an \mathcal{A}_n -algebra structure is the homotopy class

$$[\phi(\mu_n) \circ \iota] \in [\sigma_n(X), X].$$

Remark 3.9 (Existence of the object $\sigma_n(X)$). In $\mathcal{T}op$, there is no guarantee that the object $\sigma_n(X)$ will exist. However, in the category of $(\mathcal{S}p, \mathbb{S}^0, \wedge)$, the object $\sigma_n(X)$ always exists, which is, up to homotopy the desuspension of $C(\phi(\mu_n))$

$$\sigma_n(X) \simeq \Sigma^{-1}C(\phi(\mu_n)).$$

This obstruction theory yields immediate result for two-cell complexes. Since the discussion takes place in the $\mathcal{S}p$ (and not in the homotopy category $h\mathcal{S}p$), it is important to give $C\tau$ an explicit pointset model. In our category of spectra, the sphere spectrum \mathbb{S}^0 is fibrant but not cofibrant, therefore τ cannot be realized as a map from $\Sigma^{k-1}\mathbb{S}^0$ to \mathbb{S}^0 . However, τ can be realized as a map

$$\tau : \Sigma^{k-1}\mathcal{S}_{\mathbb{S}}^0 \longrightarrow \mathbb{S}^0$$

$\mathcal{S}_{\mathbb{S}}^0$ is the cofibrant replacement of \mathbb{S}^0 . We specifically choose $\mathcal{S}_{\mathbb{S}}^0$ to be the cofibrant replacement of [EKMM, Equation 1.7]. Moreover, we choose and fix non-canonical and non-coherent isomorphisms

$$\underbrace{(\mathcal{S}_{\mathbb{S}}^0 \wedge \cdots \wedge \mathcal{S}_{\mathbb{S}}^0)}_{n \text{ times}} \cong \mathcal{S}_{\mathbb{S}}^0$$

for all $n > 0$. What we gain is the isomorphism

$$(3.10) \quad (\Sigma^k \mathcal{S}_{\mathbb{S}}^0)^{\wedge n} \cong \Sigma^{kn} \mathcal{S}_{\mathbb{S}}^0$$

for all $k \geq 0$ and $n > 0$.

Notation 3.11. For efficiency of notations, we will denote $\Sigma^n \mathcal{S}_S^0$ by \mathcal{S}_S^n and $C\Sigma^n \mathcal{S}_S^0$, the cone on $\Sigma^n \mathcal{S}_S^0$, by \mathcal{D}_S^{n+1} .

The pointset model of $C\tau$ that we choose is the pushout in the diagram

$$(3.12) \quad \begin{array}{ccc} \mathcal{S}_S^{k-1} & \xrightarrow{\tau} & \mathbb{S}^0 \\ \downarrow & & \downarrow \\ \mathcal{D}_S^k & \longrightarrow & C\tau. \end{array}$$

With this model of $C\tau$ and Equation 3.10, it can be easily seen that $\mathbb{S}[\mathcal{K}^n] \wedge (C\tau)^{\wedge n}$ is related to $\phi_n(C\tau)$ via the pushout diagram

$$(3.13) \quad \begin{array}{ccc} \mathcal{S}_S^{n(k+1)-3} & \xrightarrow{\iota} & \phi_n(C\tau) \\ \downarrow & & \downarrow \\ \mathcal{D}_S^{n(k+1)-2} & \xrightarrow{\bar{i}} & \mathbb{S}[\mathcal{K}^n] \wedge (C\tau)^{\wedge n}. \end{array}$$

Therefore, $\sigma_n(C\tau) \simeq \Sigma^{n(k+1)-3} \mathbb{S}^0$ and we get:

Corollary 3.14. *Suppose $C\tau$, the cofiber of $\tau \in \pi_{k-1}(\mathbb{S}^0)$, admits a unital A_{n-1} -structure, then the obstruction to a unital A_n -structure lies in the stable homotopy group $\pi_{n(k+1)-3}(C\tau)$.*

Here are some easy applications of the result above.

Example 3.15. Oka [O] proved that $M_2(i)$ for $i \geq 2$ admits an A_3 -structure. Observe that $\pi_5(M_2(i)) = 0$ for all i . Therefore, $M_2(i)$ admits a unital A_4 -structure when $i \geq 2$.

Example 3.16. At odd primes p , the obstruction to A_n -structure on $M_p(i)$ lies in $\pi_{2n-3}(M_p(i))$. The lack of nontrivial homotopy elements at odd primes till degree $2p-4$ guarantees a unital A_{p-1} -structure on all $M_p(i)$. It is known that $M_p(1)$ does not admit a unital A_p -structure and the obstruction is precisely α_1 (see [A1, Example 3.3] for a proof of this result).

4. OBSTRUCTION THEORY FOR A_n -MAPS

Let $(\mathcal{C}, \otimes, \mathbf{I})$ be a closed symmetric monoidal category which satisfies the conditions **(C1)**, **(C2)**, **(C3)** and **(S1)** as described in Subsection 2.3.

Definition 4.1. Suppose X and Y are \mathcal{A}_n° -algebras in \mathcal{C} . A map $f : X \rightarrow Y$ is an \mathcal{A}_n° -map or a *nonunital A_n -map*, if the diagram

$$(4.2) \quad \begin{array}{ccc} \mathcal{K}^i \boxtimes X^{\otimes i} & \xrightarrow{\mu_i^X} & X \\ \mathcal{K}^i \boxtimes f^{\otimes i} \downarrow & & \downarrow f \\ \mathcal{K}^i \boxtimes Y^{\otimes i} & \xrightarrow{\mu_i^Y} & Y \end{array}$$

commutes for all $1 \leq i \leq n$. If X and Y are \mathcal{A}_n -algebras then f is an \mathcal{A}_n -map or a *unital A_n -map* if the above diagram commutes for $0 \leq i \leq n$.

Sometimes it is convenient to express the diagram in Equation 4.1 in terms of function spaces

$$\begin{array}{ccc} \mathcal{K}_+^i & \xrightarrow{\mu_i^X} & \mathcal{F}(X^{\otimes i}, X) \\ \mu_i^Y \downarrow & & \downarrow f_* \\ \mathcal{F}(Y^{\otimes i}, Y) & \xrightarrow{(f^{\otimes i})^*} & \mathcal{F}(X^{\otimes i}, Y). \end{array}$$

If X and Y admit nonunital A_n -structures and $f : X \rightarrow Y$ is a nonunital A_{n-1} -map then we have a commutative diagram

$$\begin{array}{ccc} \partial\mathcal{K}_+^n & \xrightarrow{\partial\mu_n^X} & \mathcal{F}(X^{\otimes n}, X) \\ \partial\mu_n^Y \downarrow & & \downarrow f_* \\ \mathcal{F}(Y^{\otimes n}, Y) & \xrightarrow{(f^{\otimes n})^*} & \mathcal{F}(X^{\otimes n}, Y). \end{array}$$

The most obvious, but not correct, way to build an obstruction theory is to consider the map α obtained by gluing $f_* \circ \mu_n^X$ and $(f^{\otimes n})^* \circ \mu_n^Y$ along $\partial\mathcal{K}^n$

$$\begin{array}{ccc} \partial\mathcal{K}_+^n \hookrightarrow \mathcal{K}_+^n & & \\ \downarrow & \downarrow & \searrow f_* \circ \mu_n^X \\ \mathcal{K}_+^n & \xrightarrow{\quad} & (\mathcal{K}^n \cup_{\partial\mathcal{K}^n} \mathcal{K}^n)_+ \\ & \searrow (f^{\otimes n})^* \circ \mu_n^Y & \nearrow \alpha \\ & & \mathcal{F}(X^{\otimes n}, Y). \end{array}$$

If the map α is not homotopic to a constant map then it is clear that the map f cannot be extended to a nonunital A_n -map. However, the shortcoming of this method is the following. Suppose α is homotopic to a constant, then the diagram

$$\begin{array}{ccc} \mathcal{K}_+^n & \xrightarrow{\mu_n^X} & \mathcal{F}(X^{\otimes n}, X) \\ \mu_n^Y \downarrow & & \downarrow f_* \\ \mathcal{F}(Y^{\otimes n}, Y) & \xrightarrow{(f^{\otimes n})^*} & \mathcal{F}(X^{\otimes n}, Y) \end{array}$$

commutes but only up to homotopy. Thus obstruction being trivial up to homotopy does not necessarily extend f to a nonunital A_n -map. One way to fix this problem is to relax the notion of A_n -maps to ‘homotopy A_n -maps’.

For any operad \mathcal{O} , one can define ‘homotopy \mathcal{O} -algebra map’ which is due to Boardman and Vogt [BV]. In this paper, we will restrict ourselves to the Stasheff A_n -operad.

4.1. Obstruction theory for homotopy A_n -maps. Suppose X and Y are objects in \mathcal{C} that admit nonunital A_n -structures. A *nonunital homotopy A_n -map* is a collection of maps

$$f_i : \mathcal{J}_+^i \longrightarrow \mathcal{F}(X^{\otimes i}, Y)$$

for $1 \leq i \leq n$, where \mathcal{J}^i is a polytope homeomorphic to D^{i-1} called the i -th multiplihedron in the literature. The map f_1 is equal to f and f_i for $i > 1$ is a ‘coherent’ homotopy between $(f^{\otimes i})^* \circ \mu_i^Y$ and $f_* \circ \mu_i^X$. The coherence condition will be clear once we explain what multiplihedra are. The polytope \mathcal{J}^1 is a point, $\mathcal{J}^2 = [0, 1]$ which parameterizes the homotopy

$$f_2 : \mu_2^X \circ f_* \simeq \mu_2^Y \circ (f \otimes f)^*.$$

The polytope \mathcal{J}^3 is a hexagon and the map ∂f_3 , on $\partial \mathcal{J}^3$, is determined by f_1 , f_2 and the A_3 -structures of X and Y as illustrated in the diagram below. An

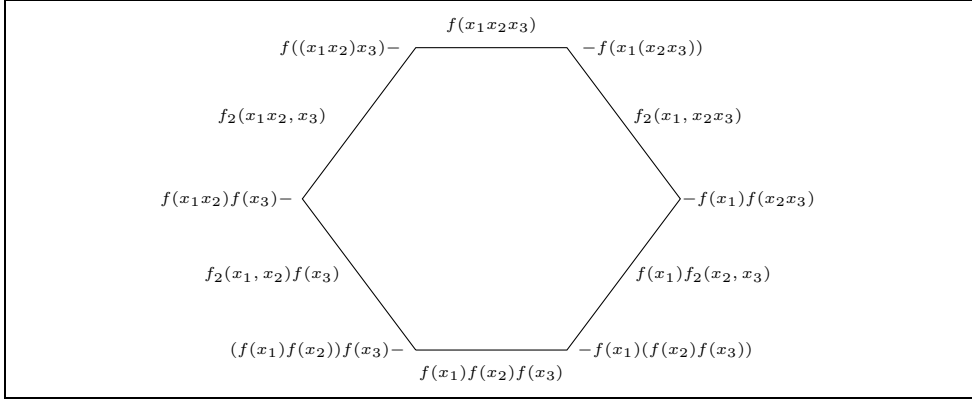


Figure 4: The polytope \mathcal{J}^3

extension of ∂f_3 to the entire hexagon \mathcal{J}^3 will be denoted by f_3 . In general, the maps $\{f_r : 1 \leq r \leq i-1\}$ along with the maps $\{\mu_r^X, \mu_r^Y : 1 \leq r \leq i\}$ determine the map ∂f_i defined on $\partial \mathcal{J}^i$. On the boundary of \mathcal{J}^i there are two disjoint cells homeomorphic to \mathcal{K}^i , one of which supports the map $(f^{\otimes i})^* \circ \mu_i^Y$ and the other supports $f_* \circ \mu_i^X$. Therefore the map f_i in some sense is a ‘homotopy with additional coherence conditions’ between the two maps. We summarize the above discussion with the following theorem.

Theorem 4.3 (Stasheff). *A nonunital homotopy A_{n-1} -map $f : X \rightarrow Y$, where X and Y admit nonunital A_n -structures, extends to a nonunital homotopy A_n -map if and only if*

$$\partial f_n : \mathcal{J}_+^n \longrightarrow \mathcal{F}(X^{\otimes n}, Y)$$

is homotopic to a constant. In other words, the obstruction to extending a nonunital homotopy A_{n-1} -map to a nonunital homotopy A_n -map is the homotopy class $[\partial f_n]$.

Multiplihedra and their connection to homotopy A_n -maps was first considered by Stasheff [STA I, II]. Boardman and Vogt [BV] expressed the full combinatorial descriptions of these multiplihedra using the language of colored operads and metric trees. In the literature, [DF, F, IM] are among other prominent articles with detailed descriptions of multiplihedra.

The above discussion can be extended to develop a unital version of homotopy A_n -maps. Let X and Y be objects in \mathcal{C} that admit unital A_n -structures. A *unital homotopy A_n -map* $f : X \rightarrow Y$, is a collection of maps

$$f_i : \mathcal{J}_+^i \longrightarrow F(X^{\otimes i}, Y)$$

for $0 \leq i \leq n$ with the convention that $\mathcal{J}^0 = *$ and $f_1 = f$, which satisfies the usual compatibility conditions.

Let $f : X \rightarrow Y$ be a unital homotopy A_{n-1} -map. A unital homotopy A_{n-1} -map is always a nonunital homotopy A_{n-1} -map. Thus f determines a map

$$\partial f_n : \partial \mathcal{J}_+^n \longrightarrow \mathcal{F}(X^{\otimes n}, Y).$$

The unital condition guarantees that ∂f_n when composed with the restriction map $\mathcal{F}(X^{\otimes n}, Y) \rightarrow \mathcal{F}(X_{[n-1]}^{\otimes n}, Y)$, extends to \mathcal{J}_+^n (compare Lemma 3.4), producing the map $f_n^{[n-1]}$ in the diagram

$$\begin{array}{ccc} \partial \mathcal{J}_+^n & \xrightarrow{\partial f_n} & \mathcal{F}(X^{\otimes n}, Y) \\ \downarrow & & \downarrow \\ \mathcal{J}_+^n & \xrightarrow{f_n^{[n-1]}} & \mathcal{F}(X_{[n-1]}^{\otimes n}, Y). \end{array}$$

Let $\kappa_n(X)$ be the pushout in the diagram

$$(4.4) \quad \begin{array}{ccc} \partial \mathcal{J}^n \boxtimes X_{[n-1]}^{\otimes n} & \longrightarrow & \partial \mathcal{J}^n \boxtimes X^{\otimes n} \\ \downarrow & & \downarrow \\ \mathcal{J}^n \boxtimes X_{[n-1]}^{\otimes n} & \longrightarrow & \kappa_n(X). \end{array}$$

The adjoint of the maps ∂f_n and $f_n^{[n-1]}$ determine the map $\kappa_n(f) : \kappa_n(X) \rightarrow Y$ in the diagram

$$(4.5) \quad \begin{array}{ccc} \partial \mathcal{J}^n \boxtimes X_{[n-1]}^{\otimes n} & \longrightarrow & \partial \mathcal{J}^n \boxtimes X^{\otimes n} \\ \downarrow & & \downarrow \\ \mathcal{J}^n \boxtimes X_{[n-1]}^{\otimes n} & \longrightarrow & \kappa_n(X) \end{array} \quad \begin{array}{c} \nearrow \partial f_n \\ \searrow \kappa_n(f) \\ \nearrow f_n^{[n-1]} \end{array} \quad \begin{array}{c} Y \\ \nearrow \end{array}$$

Suppose that there exists $\lambda_n(X) \in \text{Obj}(\mathcal{C})$ such that

$$(4.6) \quad \lambda_n(X) \xrightarrow{\gamma} \kappa_n(X) \longrightarrow \mathcal{J}^n \boxtimes X^{\otimes n}$$

is a cofiber sequence, then we can conclude:

Theorem 4.7. *Suppose $f : X \rightarrow Y$ is a unital homotopy A_{n-1} -map then the obstruction to extending f to a unital homotopy A_m -map is the homotopy class*

$$[\kappa_m(f) \circ \gamma] \in [\lambda_m(X), Y].$$

Let $X = C\tau$, the cofiber of $\tau \in \pi_{k-1}(\mathbb{S}^0)$. Upon choosing the pointset model of $C\tau$ as described in 3.12, we see that $\mathbb{S}[\mathcal{J}^n] \wedge (C\tau)^{\wedge n}$ is a pushout in the diagram

$$(4.8) \quad \begin{array}{ccc} \mathcal{S}_{\mathbb{S}}^{n(k+1)-2} & \xrightarrow{\gamma} & \kappa_n(C\tau) \\ \downarrow & & \downarrow \\ \mathcal{D}_{\mathbb{S}}^{n(k+1)-1} & \xrightarrow{\bar{\gamma}} & \mathbb{S}[\mathcal{J}^n] \wedge (C\tau)^{\wedge n}. \end{array}$$

As a result $\lambda_n(C\tau) \simeq \Sigma^{n(k+1)-2}\mathbb{S}^0$, and we get:

Corollary 4.9. *Suppose $X = C\tau$ and Y admit unital A_n -structures and there exists a unital homotopy A_{n-1} -map*

$$f : C\tau \longrightarrow Y,$$

then the obstruction to extending f to a unital homotopy A_n -map lies in the homotopy group $\pi_{n(k+1)-2}(Y)$.

4.2. Altering A_n -structures on two-cell complexes. Suppose $X = C\tau \in \mathcal{S}p$, where $\tau \in \pi_{k-1}(\mathbb{S}^0)$, admits an A_n -structure, i.e. a compatible set of maps

$$\mu_i : \mathbb{S}[\mathcal{K}^i] \wedge C\tau^{\wedge i} \longrightarrow C\tau$$

for $0 \leq i \leq n$. We will develop a method of altering the A_n -structure by altering μ_n by elements in $\pi_{n(k+1)-2}(C\tau)$. With careful analysis one can develop this notion in a more general setup. However, we restrict ourselves to two-cell complexes as that is all we need in the proof of Theorem 4.18.

In Equation 3.13, we observed that $\mathbb{S}[\mathcal{K}^n] \wedge (C\tau)^{\wedge n}$ is a pushout in the diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} & \xrightarrow{\iota} & \phi_n(C\tau) \\ \downarrow & & \downarrow \\ \mathcal{D}_{\mathbb{S}}^{n(k+1)-2} & \xrightarrow{\bar{\iota}} & \mathbb{S}[\mathcal{K}^n] \wedge (C\tau)^{\wedge n}. \end{array}$$

In Section 3, we saw that the maps μ_0, \dots, μ_{n-1} enable us to construct a map

$$\phi(\mu_n) : \phi_n(C\tau) \longrightarrow X.$$

The map μ_n is nothing but a choice of null homotopy, say H_n , of the map $\phi(\mu_n) \circ \iota$ as shown in the diagram

$$(4.10) \quad \begin{array}{ccc} \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} & \xrightarrow{\iota} & \phi_n(C\tau) \\ \downarrow & & \downarrow \\ \mathcal{D}_{\mathbb{S}}^{n(k+1)-2} & \xrightarrow{\bar{\iota}} & \mathbb{S}[\mathcal{K}^n] \wedge C\tau^{\wedge n} \end{array} \quad \begin{array}{c} \nearrow \phi(\mu_n) \\ \searrow \mu_n \\ \downarrow H_n \end{array} \quad \begin{array}{c} \\ \\ X \end{array}$$

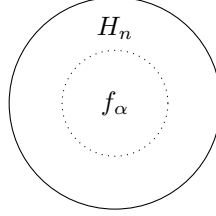
Therefore, we can alter μ_n by altering H_n by $\alpha \in \pi_{n(k+1)-2}(C\tau)$ in the following manner. Choose a map f_α in the homotopy class of $\alpha \in \pi_{n(k+1)-2}(C\tau)$. Regard f_α as a map of pairs

$$(\mathcal{D}_{\mathbb{S}}^{n(k+2)-2}, \mathcal{S}_{\mathbb{S}}^{n(k+2)-3}) \longrightarrow (C\tau, *).$$

Now define

$$H_n^\alpha : \mathcal{D}_\mathbb{S}^{n(k+1)-2} \longrightarrow C\tau$$

by concatenating f_α and H_n as depicted in the picture below.



The map H_n^α induces a map

$$\mu_n^\alpha : \mathbb{S}[\mathcal{K}^n] \wedge C\tau^{\wedge n} \longrightarrow C\tau$$

such that the maps $\{\mu_0, \dots, \mu_{n-1}, \mu_n^\alpha\}$ form a unital A_n -structure on $C\tau$. We call this process *altering the A_n -structure by α* . This construction is independent of the choice of f_α in the following sense. It is straightforward from the construction that the identity map $1 : C\tau \rightarrow C\tau$ is a unital homotopy A_{n-1} -map between the original A_n -structure on $C\tau$ and the altered one. Moreover, the obstruction to extending the identity map to a unital homotopy A_n -map is the homotopy class of f_α , which is $\alpha \in \pi_{n(k+1)-2}(\mathbb{S}^0)$.

4.3. Applications of the obstruction theory. Let $\tau \in \pi_{k-1}(\mathbb{S}_p^0)$ be a nonzero element in the stable homotopy groups of sphere completed at the prime p . For any such τ , we have a map $f : Cp\tau \rightarrow C\tau$ in $\mathcal{S}p$ which fits in the diagram

$$(4.11) \quad \begin{array}{ccccc} \mathcal{S}_\mathbb{S}^{k-1} & \xrightarrow{p\tau} & \mathbb{S}^0 & \longrightarrow & Cp\tau \\ p \downarrow & & \parallel & & \downarrow f \\ \mathcal{S}_\mathbb{S}^{k-1} & \xrightarrow{\tau} & \mathbb{S}^0 & \longrightarrow & C\tau. \end{array}$$

In particular, when $\tau = p^{i-1}$, we denote the map f by

$$f_p^i : M_p(i) \longrightarrow M_p(i-1).$$

If $M_p(i)$ and $M_p(i-1)$ admit unital A_{n-1} -structures and f_p^i extends to a unital homotopy A_{n-1} -map, then we can relate the obstruction to unital A_n -structure of $M_p(i)$ to that of $M_p(i-1)$. The technique that allows us to relate the obstruction classes can be discussed in a much general setting.

First we make the discussion for the nonunital case. Let $X, Y \in \text{Obj}(\mathcal{C})$ those admit nonunital A_{n-1} -structures and $f : X \rightarrow Y$ be a homotopy A_{n-1} -map. Recall that a homotopy A_{n-1} -map comes with the data of compatible set of maps

$$f_i : \mathcal{J}^i \boxtimes X^{\wedge i} \longrightarrow Y$$

for $1 \leq i \leq n-1$. The polytope \mathcal{J}^i has two disjoint cells homeomorphic to \mathcal{K}^n on its boundary, call them $\mathcal{K}^i[1]$ and $\mathcal{K}^i[2]$, whose purposes are to parametrize the maps $f \circ \mu_i^X$ and $\mu_i^Y \circ (f^{\otimes i})$ respectively. Thus f_i can be regarded as a coherent homotopy between these two maps. With this data one can define a map on the boundary of \mathcal{J}^n with the interior of $\mathcal{K}^n[1]$ and $\mathcal{K}^n[2]$ removed

$$\tilde{f}_n : (\partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ))_+ \longrightarrow \mathcal{F}(X^{\otimes n}, Y).$$

Since $\partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ)$ is homeomorphic to $\partial\mathcal{K}^n \times [0, 1]$, \tilde{f}_n can be regarded as a homotopy between the maps defined on $\partial\mathcal{K}^n[1]$ and $\partial\mathcal{K}^n[2]$. Thus \tilde{f}_n makes the diagram

$$\begin{array}{ccc} \partial\mathcal{K}_+^n & \xrightarrow{\partial\mu_n^X} & \mathcal{F}(X^{\otimes n}, X) \\ \partial\mu_n^Y \downarrow & & \downarrow f_* \\ \mathcal{F}(Y^{\otimes n}, Y) & \xrightarrow{(f^{\otimes n})^*} & \mathcal{F}(X^{\otimes n}, Y) \end{array}$$

commute up to homotopy. The homotopy classes $[\partial\mu_n^X]$ and $[\partial\mu_n^Y]$, are nothing but obstructions to extending nonunital A_{n-1} -structures to nonunital A_n -structures on X and Y respectively. Therefore, the above diagram can be regarded as a relation between the obstructions to nonunital A_n -structures of X and Y .

The discussion can be easily extended to the unital case. Making use of the unit maps on X and Y , one can extend the restriction of \tilde{f}_n to $X_{[n-1]}^{\otimes n}$, on the entire \mathcal{J}^n

$$(4.12) \quad \begin{array}{ccc} \partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ)_+ & \xrightarrow{\tilde{f}_n} & \mathcal{F}(X^{\otimes n}, Y) \\ \downarrow & & \downarrow \\ \mathcal{J}_+^n & \xrightarrow{\tilde{f}_n^{[n-1]}} & \mathcal{F}(X_{[n-1]}^{\otimes n}, Y). \end{array}$$

Let $\tilde{\kappa}_n(X)$ be the pushout of the diagram

$$\begin{array}{ccc} \partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ) \boxtimes X_{[n-1]}^{\otimes n-1} & \longrightarrow & \partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ) \boxtimes X^{\otimes n} \\ \downarrow & & \downarrow \\ \mathcal{J}^n \boxtimes X_{[n-1]}^{\otimes n-1} & \longrightarrow & \tilde{\kappa}_n(X). \end{array}$$

Using the adjoint of the maps of the diagram in Equation 4.12, one can construct a map $\tilde{\kappa}_n(f) : \tilde{\kappa}_n(X) \rightarrow Y$ in the diagram

$$\begin{array}{ccc} \partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ) \boxtimes X_{[n-1]}^{\otimes n-1} & \longrightarrow & \partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ) \boxtimes X^{\otimes n} \\ \downarrow & & \downarrow \\ \mathcal{J}^n \boxtimes X_{[n-1]}^{\otimes n-1} & \longrightarrow & \tilde{\kappa}_n(X) \end{array} \quad \begin{array}{c} \nearrow \tilde{f}_n \\ \searrow \tilde{\kappa}_n(f) \\ \xrightarrow{\tilde{f}_n^{[n-1]}} Y. \end{array}$$

Since $\partial\mathcal{J}^n \setminus (\mathcal{K}^n[1]^\circ \sqcup \mathcal{K}^n[2]^\circ) \cong [0, 1] \times \partial\mathcal{K}^n$, $\mathcal{J}^n \cong [0, 1] \times K^n$ and $[0, 1] \boxtimes (_)$ preserves pushout, it follows that

$$\tilde{\kappa}_n(X) \cong [0, 1] \boxtimes \phi_n(X),$$

where $\phi_n(X)$ is the object in \mathcal{C} as defined in Equation 3.5. The composite map

$$[0, 1] \boxtimes \sigma_n(X) \longrightarrow [0, 1] \boxtimes \phi_n(X) \cong \tilde{\kappa}_n(X) \xrightarrow{\tilde{\kappa}_n(f)} Y$$

is the homotopy that makes the diagram

$$(4.13) \quad \begin{array}{ccc} \sigma_n(X) & \xrightarrow{\phi(\mu_n^X) \circ \iota} & X \\ \downarrow & & \downarrow \\ \sigma_n(Y) & \xrightarrow{\phi(\mu_n^Y) \circ \iota} & Y \end{array}$$

homotopy commutative. The homotopy classes $[\phi(\mu_n^X) \circ \iota]$ and $[\phi(\mu_n^Y) \circ \iota]$ are the obstruction to extending unital A_{n-1} -structure to unital A_n -structure on X and Y respectively. Therefore, the above diagram can be regarded as a relation between these two obstruction classes.

Notation 4.14. To simplify notations, $\theta_n(X)$ denote $[\phi(\mu_n^X) \circ \iota]$, the homotopy class of obstructions to the unital A_n -structure on X .

When $X = Cp\tau$ and $Y = C\tau$ in the category \mathcal{Sp} , where $\tau \in \pi_{k-1}(\mathbb{S}_p^0)$, by Corollary 3.14 we have

$$\sigma_n(X) \cong \sigma_n(Y) \cong \mathcal{S}_{\mathbb{S}}^{n(k+1)-3}.$$

Suppose that the map

$$f : Cp\tau \longrightarrow C\tau$$

defined in Equation 4.11 extends to a unital homotopy A_{n-1} -map, the diagram of Equation 4.13 is precisely

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} & \xrightarrow{\theta_n(Cp\tau)} & Cp\tau \\ p^n \downarrow & & \downarrow f \\ \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} & \xrightarrow{\theta_n(C\tau)} & C\tau, \end{array}$$

relating the obstruction to unital A_n -structures of $Cp\tau$ to that of $C\tau$. The left vertical map in the above diagram is multiplication by p^n map because the attaching map of the top cell of $Cp\tau^{\wedge n}$ and $C\tau^{\wedge n}$ is related to the p^n map

$$\begin{array}{ccccc} \mathcal{S}_{\mathbb{S}}^{nk-1} & \longrightarrow & Cp\tau_{[n-1]}^{\wedge n} & \longrightarrow & Cp\tau^{\wedge n} \\ p^n \downarrow & & f_{[n-1]}^{\wedge n} \downarrow & & \downarrow f^{\wedge n} \\ \mathcal{S}_{\mathbb{S}}^{nk-1} & \longrightarrow & Cp\tau_{[n-1]}^{\wedge n} & \longrightarrow & Cp\tau^{\wedge n}. \end{array}$$

The relation between the obstruction to A_n -structures on $Cp\tau$ and $C\tau$ can be made more explicit if the obstruction classes factor through the bottom cell. Under such circumstances we get

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} & \xrightarrow{p^n} & \mathbb{S}^{n(k+1)-3} \\ \downarrow & & \downarrow \\ \theta_n(Cp\tau) \left(\begin{array}{c} \mathbb{S}^0 \\ \parallel \\ \mathbb{S}^0 \end{array} \right) & & \theta_n(C\tau) \left(\begin{array}{c} \mathbb{S}^0 \\ \parallel \\ \mathbb{S}^0 \end{array} \right) \\ \downarrow & & \downarrow \\ Cp\tau & \xrightarrow{f} & C\tau, \end{array}$$

i.e. $\theta_n(Cp\tau) = p^n \theta_n(X)$ in $\pi_*(\mathbb{S}^0)$. We record this observation as the following lemma.

Lemma 4.15. *Let $\tau \in \pi_{k-1}(\mathbb{S}^0)$. Suppose $C\tau$ and $Cp\tau$ admit unital A_{n-1} -structures and let the map $f : Cp\tau \rightarrow C\tau$ of Equation 4.11 be a unital homotopy A_{n-1} -map. Let $\theta_n(C\tau) \in \pi_{n(k+1)-3}(C\tau)$ and $\theta_n(Cp\tau) \in \pi_{n(k+1)-3}(Cp\tau)$ be the obstructions to the unital A_n -structure of $C\tau$ and $Cp\tau$, respectively. If both $\theta_n(C\tau)$ and $\theta_n(Cp\tau)$ factor through the unit map from \mathbb{S}^0 , then*

$$\theta_n(Cp\tau) = p^n \theta_n(X).$$

Remark 4.16. If $\theta_n(C\tau)$ does not factor through the bottom cell, i.e. the composition

$$\mathcal{S}_{\mathbb{S}}^{k(n+1)-3} \xrightarrow{\theta_n(C\tau)} Cp\tau \xrightarrow{\text{pinch}} \mathcal{S}_{\mathbb{S}}^k$$

is not null homotopic, then we may run into a degenerate situation in the following manner. Suppose $e \in \pi_*(\mathbb{S}^0)$ such that $p^n e = 0$ and $\theta_n(C\tau) \circ \text{pinch} = e$, then from the diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} & \xrightarrow{p^n} & \mathcal{S}_{\mathbb{S}}^{n(k+1)-3} \\ \downarrow \theta_n(Cp\tau) & & \downarrow \theta_n(C\tau) \\ e' \left(Cp\tau \right) & \xrightarrow{f} & C\tau \right) e \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathbb{S}}^k & \xrightarrow{p} & \mathcal{S}_{\mathbb{S}}^k \end{array}$$

we observe that e' can be any element in $\pi_{n(k+1)-4}(\mathbb{S}^0)$, which satisfies $pe' = 0$, (including the possibility that $e' = 0$). Therefore, there is no strict relation between the obstructions to unital A_n -structures of $C\tau$ and $Cp\tau$ unless the obstructions factor through the bottom cell.

Now we apply the above discussion to the case when $\tau = p^k$ for an odd prime p . First we recall the homotopy groups of $M_p(k)$ in the range $0 \leq n \leq 2(p^2 - p - 1)$. One can compute the homotopy groups of $M_p(k)$ in this range using the knowledge of homotopy groups of spheres and long exact sequence

$$\dots \longrightarrow \pi_n(\mathbb{S}_p^0) \xrightarrow{p^k} \pi_n(\mathbb{S}_p^0) \xrightarrow{\mu_0} \pi_n(M_p(k)) \xrightarrow{\text{pinch}} \pi_n(\Sigma \mathbb{S}_p^0) \longrightarrow \dots$$

The groups $\pi_n(\mathbb{S}_p^0)$ are generated by the Greek letter elements α_i for $1 \leq i \leq p-1$, $\alpha_{\frac{p}{2}}$ and β_1

$$\pi_n(\mathbb{S}_p^0) = \begin{cases} \mathbb{Z}_p & n = 0 \\ \mathbb{Z}/p\langle \alpha_i \rangle & n = 2i(p-1) - 1 \text{ for } 1 \leq i \leq p-1 \\ \mathbb{Z}/p\langle \beta_1 \rangle & n = 2p^2 - 2p - 2 \\ \mathbb{Z}/p^2\langle \alpha_{\frac{p}{2}} \rangle & n = 2p^2 - 2p - 1 \\ 0 & \text{otherwise} \end{cases}$$

when $0 \leq n \leq 2(p^2 - p - 1)$. All these generators map nontrivially to $\pi_*(M_p(k))$ under the unit map μ_0 and we abusively use the same notation to denote the image of these generators. In this range, $\pi_*(M_p(k))$ also has generators which map to

Greek letter elements in $\pi_*(S_p^0)$ under the pinch map. We denote these generators by $\alpha_i^{(k)}$ for $1 \leq i \leq p$, $\alpha_{\frac{p}{2}}^{(k)}$ and $\beta_1^{(k)}$. Thus we have,

$$(4.17) \quad \pi_n(M_p(k)) = \begin{cases} \mathbb{Z}/p^k & t = 0 \\ \mathbb{Z}/p\langle\alpha_i\rangle & n = 2i(p-1) - 1 \text{ for } 1 \leq i \leq p-1 \\ \mathbb{Z}/p\langle\alpha_i^{(k)}\rangle & n = 2i(p-1) \text{ for } 1 \leq i \leq p-1 \\ \mathbb{Z}/p\langle\beta_1\rangle & n = 2p^2 - 2p - 2 \\ \mathbb{Z}/p^\epsilon\langle\alpha_{\frac{p}{2}}\rangle \times \mathbb{Z}/p\langle\beta_1^{(k)}\rangle & n = 2p^2 - 2p - 1 \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq n \leq 2(p^2 - p - 1)$, $\epsilon = 1$ if $k = 1$ and $\epsilon = 2$ when $k > 1$.

The following is the best result that we can obtain using the obstruction theory techniques that we have developed so far.

Theorem 4.18. *For an odd prime p , $M_p(k)$ admits a unital $A_{2k(p-1)}$ -structure for $1 \leq k \leq p$. For $k \geq p$, $M_p(k)$ admits a unital $A_{2p(p-1)}$ -structure.*

Before proving Theorem 4.18, we explain the effect of altering the A_n -structures of $M_p(k)$ or $M_p(k-1)$ on the obstruction class for extending

$$f_p^k : M_p(k) \longrightarrow M_p(k-1)$$

to a unital homotopy A_n -map.

Notation 4.19. To avoid cumbersome notations we abbreviate

- $\phi_n(M_p(k))$ to $\phi_n(k)$,
- $\kappa_n(M_p(k))$ to $\kappa_n(k)$,
- $\sigma_n(M_p(k))$ to $\sigma_n(k)$, and,
- $\lambda_n(M_p(k))$ to $\lambda_n(k)$.

Notation 4.20. If $M_p(k)$ admits a unital A_n -structure, then

$$\mu_i(r) : \mathbb{S}[\mathcal{K}^i] \wedge M_p(r)^{\wedge i} \longrightarrow M_p(r)$$

will denote the ‘ i -fold multiplication map’ and

$$H_i(r) : C\sigma_i(r) \longrightarrow M_p(r)$$

will denote the ‘null-homotopy’ associated to the $\mu_i(r)$, as shown in Equation 4.10.

Suppose that $M_p(k)$ and $M_p(k-1)$ admit unital A_n -structures. Moreover, assume that the map

$$f_p^k : M_p(k) \longrightarrow M_p(k-1)$$

extends to a unital homotopy A_{n-1} -map.

Notation 4.21. Let the map $\kappa_n(f_p^k) \circ \gamma$, whose homotopy class is the obstruction to extending f_p^k to a unital homotopy A_n -map (see Theorem 4.7), be denoted by

$$\psi_n(k) : \lambda_n(k) \longrightarrow M_p(k-1).$$

The effect of altering the A_n -structure of $M_p(k)$ or $M_p(k-1)$ on the homotopy class $[\psi_n(k)]$ can be understood, once we know the relation between the maps

$$\gamma : \lambda_n(k) \longrightarrow \kappa_n(k)$$

as in Equation 4.22 and

$$\iota : \sigma_n(k) \longrightarrow \phi_n(k)$$

as in Equation 3.7. From Equation 3.13, we know that

$$\sigma_n(k) \cong \sigma_n(k-1) \cong \mathcal{S}_{\mathbb{S}}^{2n-3}$$

and from Equation 4.8 we know that

$$\lambda_n(k) \cong \lambda_n(k-1) \cong \mathcal{S}_{\mathbb{S}}^{2n-2}.$$

Making use of the fact that $\mathcal{J}^n \cong \mathcal{K}^n \times [0, 1]$ in the construction of $\kappa_n(k)$ (see Equation 4.4), one can deduce

$$\kappa_n(k) \cong \mathbb{S}[\mathcal{K}^n] \wedge M_p(k)^{\wedge n} \cup_{\phi_n(k)} \mathbb{S}[[0, 1]] \wedge \phi_n(k) \cup_{\phi_n(k)} \mathbb{S}[\mathcal{K}^n] \wedge M_p(k)^{\wedge n}.$$

As a result we have

$$(4.22) \quad \lambda_n(k) \cong C \sigma_n(k) \cup_{\sigma_n(k)} \mathbb{S}[[0, 1]] \wedge \sigma_n(k) \cup_{\sigma_n(k)} C \sigma_n(k) \simeq \Sigma \sigma_n(k).$$

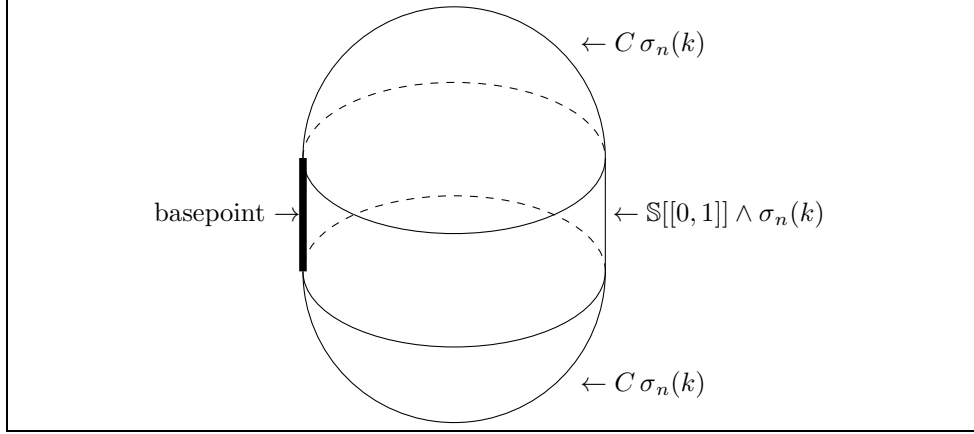


Figure 5: A diagrammatic representation of $\lambda_n(k)$ as described in Equation 4.22

It is easy to check that

$$\gamma = \tilde{\iota} \cup_{\iota} \mathbb{S}[[0, 1]] \wedge \iota \cup_{\iota} \tilde{\iota} : \lambda_n(k) \longrightarrow \kappa_n(k)$$

where $\tilde{\iota}$ is the map as depicted in Equation 4.10 when specialized to $M_p(k)$.

Lemma 4.23. *Let $M_p(k)$ and $M_p(k-1)$ admit unital A_n -structures. Suppose that*

$$f_p^k : M_p(k) \longrightarrow M_p(k-1)$$

extend to a unital homotopy A_{n-1} -map and $[\psi_n(k)]$ denote the obstruction to extending f_p^k to a unital homotopy A_n -map. Then,

- (i) *altering the unital A_n -structure of $M_p(k-1)$ by $\rho \in \pi_{2n-2}(M_p(k-1))$ changes*

$$[\psi_n(k)] \rightsquigarrow [\psi_n(k)] - \rho,$$

and,

- (ii) *altering the unital A_n -structure of $M_p(k)$ by $v \in \pi_{2n-2}(M_p(k))$ changes*

$$[\psi_n(k)] \rightsquigarrow [\psi_n(k)] - (f_p^k)_*(v).$$

Proof. Let the two copies of $C \sigma_n(k) \cong \mathcal{D}_{\mathbb{S}}^{2n-2}$ in $\lambda_n(k) \cong \mathcal{S}_{\mathbb{S}}^{2n-2}$ be denoted by $D(k)_+$ and $D(k)_-$. The map $\psi_n(k)$ when restricted to $D(k)_+$ is the composite

$$D(k)_+ \xrightarrow{\cong} D(k-1)_+ \xrightarrow{H_n(k-1)} M_p(k-1).$$

Therefore, if we alter the A_n -structure on $M_p(k-1)$ by an element $\rho \in \pi_{2n-2}(M_p(k-1))$, then $\psi_n(k)$ changes to a map that is obtained by concatenating $\psi_n(k)$ with ρ . On the other hand, the map $\psi_n(k)$ when restricted to $D(k)_-$ is precisely the composite

$$D(k)_- \xrightarrow{H_n(k)} M_p(k) \xrightarrow{f_p^k} M_p(k-1).$$

Therefore, if we alter the A_n -structure on $M_p(k)$ by an element, say $v \in \pi_{2n-2}(M_p(k-1))$, then the map $\psi_n(k)$ changes to a map obtained by concatenating $\psi_n(k)$ with $f_p^k \circ v$. The result follows from the above observations. \square

Proof of Theorem 4.18. We will first prove the following claim using induction on k :

Claim 1. The spectrum $M_p(k)$ admits a unital $A_{2k(p-1)}$ -structure. Moreover, all possible alternates, obtained by altering this structure by

$$c\alpha_j^{(k)} \in \pi_{2j(p-1)}(M_p(k))$$

for $j < k$, also extend to unital $A_{2k(p-1)}$ -structure.

Throughout this proof we repeatedly use the facts

- the obstruction to unital A_n -structure on $M_p(i)$ lives in $\pi_{2n-3}(M_p(i))$ (see Corollary 3.14),
- and the obstruction to

$$f_p^k : M_p(k) \longrightarrow M_p(k-1)$$

being unital homotopy A_n -map lives in $\pi_{2n-2}(M_p(k-1))$ (see Corollary 4.9).

We have shown that $M_p(1)$ admits a unital A_{p-1} -structure (see Example 3.16). Since $\pi_{2n-2}(M_p(1)) = 0$ for $2 \leq n \leq p-1$, there is no scope for altering the A_n -structure on $M_p(1)$. Thus Claim 1 is true for $n = 1$.

Clearly, $M_p(2)$ admits a unital A_{p-1} -structure as $\pi_{2i-3}(M_p(2)) = 0$ for $0 \leq i \leq p-1$. The map

$$f_p^2 : M_p(2) \longrightarrow M_p(1)$$

is a unital homotopy A_{p-1} -map as

$$\pi_{2i-2}(M_p(1)) = 0$$

for $2 \leq i \leq p-1$. The element $\alpha_1 \in \pi_{2p-3}(M_p(1))$, the obstruction to unital A_p -structure of $M_p(1)$, factors through the unit map

$$\mu_0(2) : \mathbb{S}^0 \longrightarrow M_p(2)$$

as the unit map is simply the inclusion of the bottom-cell. Therefore, we can apply Lemma 4.15 to see that the obstruction to a unital A_p -structure

$$p^p \alpha_1 = 0$$

is trivial (see Equation 4.17). Thus, $M_p(2)$ admits a unital A_p -structure. Note that,

$$\pi_{2i-3}(M_p(2)) = 0$$

for $p+1 \leq i \leq 2(p-1)$. Therefore, the A_p -structure and all its alternates (obtained by altering the A_p -structure by $c\alpha_1^{(2)} \in \pi_{2n-2}(M_p(2))$), extend to unital $A_{2(p-1)}$ -structures. Therefore Claim 1 is true for $k = 2$.

We prove the remaining cases by induction on k . So assume that Claim 1 is true for $M_p(k-1)$ where $2 < k \leq p$. Since

$$\pi_{2i-3}(M_p(k)) = 0$$

for $2 \leq i \leq p-1$, $M_p(k)$ admits a unital A_{p-1} -structure. Moreover, the map f_p^k can be extended to a unital homotopy A_{p-1} -map as

$$\pi_{2p-2}(M_p(k-1)) = 0$$

for $2 \leq k \leq p-1$. Thus we can apply Lemma 4.15 to see that $M_p(k)$ admits a unital A_p -structure. This does not mean

$$f_p^k : M_p(k) \longrightarrow M_p(k-1)$$

is a unital homotopy A_p -map. In fact, that the obstruction to extending f_p^k to a unital homotopy A_p -map, $\psi_p(k)$, could be

$$c\alpha_1^{(k-1)} \in \pi_{2p-2}(M_p(k-1))$$

where $c \neq 0$. If so, we alter the A_p -structure of $M_p(k-1)$ by $c\alpha_1^{(k-1)}$. By Lemma 4.23, the obstruction class $[\psi_p(k)]$ changes $c\alpha_1^{(k-1)} \rightsquigarrow 0$. Thus the map f_p^k extends to a unital homotopy A_p -map. By induction hypothesis, the alternate A_p -structure on $M_p(k-1)$ can be extended to unital $A_{(k-1)(p-1)}$ -structure. Therefore, we can continue extending f_p^k as a unital homotopy A_n -map in this manner. There are potential obstructions of the form

$$c\alpha_j^{(k-1)} \in \pi_{2j(p-1)}(M_p(k-1))$$

to extending f_p^k to unital homotopy $A_{j(p-1)+1}$ -map for each value of j , where $1 \leq j \leq k-2$. Whenever $c \neq 0$, we alter the $A_{j(p-1)+1}$ -structure of $M_p(k-1)$ by $c\alpha_j^{(k-1)}$ which, by Lemma 4.23, changes $[\psi_{j(p-1)+1}(k)]$

$$c\alpha_j^{(k-1)} \rightsquigarrow 0.$$

Thus, we can extend f_p^k to a unital homotopy $A_{(k-1)(p-1)}$ -map as $M_p(k-1)$ admits $A_{(k-1)(p-1)}$ -structure.

The inductive hypothesis guarantees $A_{(k-1)(p-1)}$ -structure on $M_p(k-1)$. The obstruction class for $A_{(k-1)(p-1)+1}$ -structure may be nontrivial and is of the form

$$c\alpha_{k-1} \in \pi_{2(k-1)(p-1)-1}(M_p(k-1))$$

as $k-1 < p$. Therefore it factors through the unit map. Now we can apply Lemma 4.15 to see that the obstruction to $M_p(k)$ admitting a unital $A_{(k-1)(p-1)+1}$ -structure is

$$p^{(k-1)(p-1)+1} \cdot c\alpha_{k-1}$$

which is trivial (see Equation 4.17). Since, $\pi_{2i-3}(M_p(k)) = 0$ for

$$(k-1)(p-1) + 2 \leq i \leq k(p-1),$$

$M_p(k)$ admits unital $A_{2k(p-1)}$ -structures.

To complete the inductive argument, we need to show that all possible alternates of the A_n -structure of $M_p(k)$, for $n < 2k(p-1)$, can be extended to a unital $A_{2k(p-1)}$ -structure. It is possible to alter the A_n -structure of $M_p(k)$, when $n = 2j(p-1)$, where $j \leq k$, by elements $c\alpha_j^{(k)} \in \pi_{2n-2}(M_p(k))$. By Lemma 4.23, the effect of such alteration on the class $[\psi_n(k)]$ is trivial as

$$f_p^k(\alpha_j^{(k)}) = 0.$$

Since, altering the A_n -structures of $M_p(k)$ by $c\alpha_j^{(k)}$ has no effect on the obstruction class, the same inductive argument is applicable to all the alternates of the A_n -structure of $M_p(k)$. Therefore, we can conclude that all of them extends to a unital $A_{k(p-1)}$ -structure.

The inductive method breaks down when $k = p+1$ as the obstruction to unital $A_{p(p-1)+1}$ -structure on $M_p(p)$ can be of the form

$$c\alpha_{\frac{p}{2}} + t\beta_1^{(p)} \in \pi_{2p^2-2p-1}(M_p(p-1))$$

with $t \neq 0$. In that case, Lemma 4.15 is not applicable (see Remark 4.16) as the obstruction elements may not factor through the unit map. However, one can carry out the same argument for $k > p$ only to conclude that $M_p(k)$ admits a unital $A_{p(p-1)}$ -structure. \square

4.4. Obstruction theory for A_n -maps in $\mathcal{T}op$. In the category $\mathcal{T}op$, Stasheff developed an obstruction theory for homotopy A_n -maps, by constructing what is called a truncated bar complex for \mathcal{A}_n -algebras. The category $\mathcal{T}op$ has the advantage that for any object $X \in \mathcal{T}op$, there is a ‘counit’ map $X \rightarrow *$ as $*$ is the terminal object. The construction of the truncated bar complex is heavily reliant on this fact.

As a warm up, we recall in brief the construction of a bar complex for a strictly associative monoid in $\mathcal{T}op$. By a strictly associative monoid in $\mathcal{T}op$, we mean a topological space H with a unit map $\iota : * \rightarrow H$ and a multiplication map

$$\mu : H \times H \longrightarrow H$$

which is compatible with the unit map, i.e. $\mu \circ (\iota \times 1_H) = 1_H = \mu \circ (1_H \times \iota)$, and is strictly associative, i.e. $\mu \circ (\mu \times 1_H) = \mu \circ (1_H \times \mu)$. A left H -module M is an object in $\mathcal{T}op$ with a map

$$m : H \times M \longrightarrow M$$

which satisfies the usual conditions. Similarly, a right H -module M is an object in $\mathcal{T}op$ with a map

$$n : N \times H \longrightarrow N$$

satisfying the usual conditions.

Let Δ be the category of finite, nonempty, totally ordered sets with order preserving maps as morphisms. Let $sk(\Delta)$ denote the skeleton category of Δ . Objects of $sk(\Delta)$ are finite ordinals, we denote the ordinal $n+1$ by $[n]$ or $\{0 < \dots < n\}$. Given a strictly associative monoid H , a left H -module M and right H -module N , we can define a functor

$$\mathcal{B}(M, H, N) : sk(\Delta)^{op} \longrightarrow \mathcal{T}op$$

where $[n] \mapsto M \times H^n \times N$. On the other hand, we have a functor

$$|\Delta| : sk(\Delta) \longrightarrow \mathcal{T}op$$

such that $[n] \mapsto \Delta(n)$, where $\Delta(n)$ is the geometric n -simplex. The two sided bar-complex $B(M, H, N)$ is the coend of the functor $\mathcal{B}(M, H, N) \times |\Delta|$ or equivalently the quotient space

$$B(M, H, N) = \coprod \Delta(n) \times (M \times H^n \times N) / \sim$$

where \sim is the usual identification expressed in terms of face and degeneracy maps. We define

$$BH = B(*, H, *)$$

as the bar complex of H .

For an \mathcal{A}_n -algebra H in $\mathcal{T}op$, a *right* \mathcal{A}_k H -module M , is an object in $\mathcal{T}op$ with maps

$$f_r : \mathcal{K}^{r+1} \times M \times H^{\times r} \longrightarrow M$$

for $1 \leq r \leq k$, satisfying the usual compatibility conditions with the higher order multiplication of H . Similarly, a *left* \mathcal{A}_k H -module N is an object in $\mathcal{T}op$ with maps

$$g_r : \mathcal{K}^{r+1} \times H^{\times r} \times N \longrightarrow N$$

for $0 \leq r \leq k$, satisfying similar compatibility conditions. The two-sided bar construction described above, does not make sense because $\mathcal{B}(M, H, N)$ fails to be a functor when the multiplication is not strictly associative. However, this issue can be resolved. Roughly speaking, the idea is to inflate the morphism classes between two objects in $sk(\Delta)^{op}$ to accommodate A_n -structures. More precisely, we enrich the category $sk(\Delta)^{op}$ over $\mathcal{T}op$ using the operad \mathcal{A}_∞ by setting the morphism class between $[l]$ and $[k]$ as the topological space

$$\bigsqcup_{f:[l] \rightarrow [k]} \mathcal{A}_\infty[f],$$

where $\mathcal{A}_\infty[f] = \prod_{i \in [k]} \mathcal{A}_\infty(f^{-1}(i))$. Denote the resultant category by Δ_∞^{op} . For an \mathcal{A}_∞ -algebra H , it is straightforward to verify that

$$\mathcal{B}_\infty(M, H, N) : \Delta_\infty^{op} \longrightarrow \mathcal{T}op$$

sending $[n] \mapsto (M \times H^n \times N)$ is indeed a functor. For brevity, let us denote $(\Delta_\infty^{op})^{op}$ by Δ_∞ . There is also a canonical functor

$$|\mathcal{K}| : \Delta_\infty \longrightarrow \mathcal{T}op$$

where $[n] \rightarrow \mathcal{K}^{n+2}$.

Definition 4.24. For an \mathcal{A}_∞ -algebra H , a right \mathcal{A}_∞ H -module M and a left \mathcal{A}_∞ H -module N , the two-sided bar complex $B(M, H, N)$ is the coend

$$B(M, H, N) = \int^{\Delta_\infty} \mathcal{B}_\infty(M, H, N) \times |\mathcal{K}|.$$

Definition 4.25. For an \mathcal{A}_∞ -algebra H , the bar complex of H is the topological space

$$BH = B(*, H, *)$$

Remark 4.26 (Abuse of the notation $B(M, H, N)$). We intentionally used the same notation for the two sided bar-complex $B(M, H, N)$, when H is a strictly associative monoid and when H is a \mathcal{A}_∞ -algebra. This is because a strictly associative monoid in $\mathcal{T}op$ is automatically an \mathcal{A}_∞ -algebra and the two different bar constructions yield the same space up to homotopy.

These were originally constructed by Stasheff [STA I, II]. Stasheff's bar construction for \mathcal{A}_∞ -algebras are explained in details in [A1].

For an \mathcal{A}_n -algebra H where $n < \infty$, one can only construct a truncated version of the bar complex, called the n -truncated bar complex. Let $\Delta_{\leq n}$ be the full subcategory of Δ_∞ with objects $[k]$ for $0 \leq k \leq n$. For an \mathcal{A}_n -algebra H , a right \mathcal{A}_n H -module M and a left \mathcal{A}_n H -module N let

$$\mathcal{B}_n(M, H, N) : \Delta_n^{op} \longrightarrow \mathcal{T}op$$

be the functor that sends $[k] \rightarrow M \times H^k \times N$. We also have a functor

$$|\mathcal{K}^n| : \Delta_{\leq n} \longrightarrow \mathcal{T}op$$

such that $[k] \mapsto \mathcal{K}^{k+2}$.

Definition 4.27. For an \mathcal{A}_n -algebra H , a right \mathcal{A}_n H -module M and a left \mathcal{A}_n H -module N the two-sided bar complex $B(M, H, N)$ is the coend

$$B_n(M, H, N) = \int^{\Delta_n} \mathcal{B}_n(M, H, N) \times |\mathcal{K}^n|.$$

The space $B_n(M, H, N)$ is a quotient of the space

$$\coprod_{0 \leq k \leq n} \mathcal{K}^{k+2} \times M \times H^k \times N.$$

Definition 4.28. For an \mathcal{A}_∞ -algebra H the bar complex of H is the topological space

$$B_n H = B_n(*, H, *)$$

Example 4.29. We know that S^1 has a strict associative multiplication when it is thought of as unit length vectors on the complex plane. The n -truncated bar complex $B_n S^1$ is homotopy equivalent to the projective space $\mathbb{C}P^n$.

Remark 4.30. For any \mathcal{A}_n -algebra H in $\mathcal{T}op$, Stasheff called the space $B_n H$ as the n -th projective space and denoted it by $HP(n)$.

The following theorem due to Stasheff (see [STA, Theorem 8.4]) is a tool to detect unital A_n -maps.

Theorem 4.31 (Stasheff). *A map $f : X \rightarrow Y$, where X and Y are strictly associative monoids in $\mathcal{T}op$, extends to a unital homotopy A_n -map if and only if the map $\Sigma f : \Sigma X \rightarrow \Sigma Y$ extends to a map*

$$B_n f : B_n X \longrightarrow B_n Y$$

where $B_n X$ and $B_n Y$ are n -truncated bar complexes of X and Y respectively.

Remark 4.32. The conclusions of Theorem 4.31 hold if X and Y are \mathcal{A}_∞ algebras in $\mathcal{T}op$. However, for the proof of Main Theorem 1 we only need the conclusions for strictly associative monoids.

Work of Boardman and Vogt [BV, Chapter 4] provides a technique to replace a homotopy A_n -map (unital or nonunital) by an A_n -map in the homotopy category of \mathcal{A}_n -algebras. Given a topological operad \mathcal{O} , they make two important constructions,

- an endofunctor W from the category of topological operads to itself such that there is a natural map of operads

$$w : W\mathcal{O} \longrightarrow \mathcal{O}$$

which is a weak equivalence, and,

- a functor $U : \mathcal{T}op[\mathcal{O}] \rightarrow \mathcal{T}op[\mathcal{O}]$, where $\mathcal{T}op[\mathcal{O}]$ is the category of \mathcal{O} -algebras in $\mathcal{T}op$, such that there is a natural map

$$u : X \longrightarrow UX$$

which is a weak equivalence and can be extended to a ‘homotopy \mathcal{O} -map’ (see Remark 4.34). We will call this map the *universal homotopy \mathcal{O} -map*.

These constructions conspire to give us the following theorem, which is essentially a special case of [BV, Theorem 4.23(c)].

Theorem 4.33. *Let X and Y be \mathcal{A}_n -algebras and $u : X \rightarrow UX$ be the universal unital homotopy \mathcal{A}_n -map. Then for any unital homotopy \mathcal{A}_n -map $f : X \rightarrow Y$ has a unique factorization*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ u \downarrow \simeq & & \parallel \\ UX & \xrightarrow{h} & Y \end{array}$$

such that h is a $W\mathcal{A}_n$ -map.

The terminologies used in [BV] are significantly different from the ones used in this paper. This can be a potential source of confusion. Hence, in the following remark, we provide a dictionary between the language in [BV] and the language used in this paper.

Remark 4.34. In [BV] authors study K -colored topological algebraic theories \mathcal{B} for a finite set K called colors [BV, Definition 2.3]. For any K colored theory \mathcal{B} , they define \mathcal{B} -spaces, \mathcal{B} -homomorphisms, homotopy \mathcal{B} -homomorphisms which are abbreviated as \mathcal{B} -maps [BV, Definition 4.1] and homotopy homogeneous \mathcal{B} -homomorphisms which are abbreviated as $h\mathcal{B}$ -maps [BV, Definition 4.2]. An operad \mathcal{B} in our language is a topological algebraic theory with one color (i.e. $K = \{*\}$) in their language, a \mathcal{B} -algebra in our language is a \mathcal{B} -space in their language and a \mathcal{B} -map in our language is a \mathcal{B} -homomorphism in their language. If we were to define a homotopy \mathcal{B} -map between two \mathcal{B} -algebras in our language, it would have been equivalent to the definition of homotopy homogeneous \mathcal{B} -map, i.e. an $h\mathcal{B}$ -map in their language. Specifically, when $\mathcal{B} = \mathcal{A}_n$, ‘homotopy \mathcal{A}_n -map’ which we call homotopy unital \mathcal{A}_n -map, is an $h\mathcal{A}_n$ -map in the language used in [BV]. This can be deduced from the discussions in [BV, Chapter 1.3] and [BV, Example 2.56].

5. THE PROOF OF THE MAIN THEOREM

The first step towards proving Main Theorem 1, is to obtain the Moore spectrum $M_p(i)$ as a Thom spectrum. We begin by describing a general construction of Thom spectra as described in [ABGHR]. This is a nontechnical sketch that presents only the gist of the construction and avoids some of the hard technical work of [ABGHR].

At this point we need the zeroth space functor, which always represents the underlying infinite loop space for this category of spectra, as the right adjoint to $\mathbb{S}[\]$. Therefore, we diverge from $\mathcal{T}op$ and work in the category of $*$ -modules

\mathcal{M} instead (see Warning 2.18). The category $(\mathcal{M}, *, \times_{\mathcal{L}})$ is a closed symmetric monoidal category that enjoys a pair of functors

$$(5.1) \quad L : \mathcal{T}op \rightleftarrows \mathcal{M} : F$$

which are a part of Quillen equivalence between these two categories. The [EKMM] category of S -module admits a loop-suspension adjunction

$$(5.2) \quad \mathbb{S}[\] : \mathcal{M} \rightleftarrows \mathcal{S}p : \Omega^\infty.$$

A detailed exposition can be found in [ABGHR, Section 3.1].

Remark 5.3. The functor $\Omega^\infty : \mathcal{S}p \rightarrow \mathcal{M}$ is weakly equivalent but not known to be isomorphic to the infinite loop space functor of [LMS] when applied to the underlying Lewis-May-Steinberger spectrum (see [Lind]). In [ABGHR], the authors denote the loop-suspension adjunction of Equation 5.2 using the symbols

$$\Sigma_{\mathbb{L}+}^\infty : \mathcal{M} \rightleftarrows \mathcal{S}p : \Omega_S^\infty.$$

They denote the classical infinite loop space functor on [LMS] category of spectra by Ω^∞ .

For a ring spectrum R , define $GL_1(R)$ to be the pullback

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array}$$

In other words, $GL_1(R)$ is the collection of components of $\Omega^\infty R$ that correspond to the units of $\pi_0(R)$. Similarly, for any subgroup H of $\pi_0(R)^\times$, define $H(R)$ as the pullback

$$\begin{array}{ccc} H(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ H & \longrightarrow & \pi_0(R). \end{array}$$

When, H is the trivial subgroup, $H(R)$ is denoted by $SL_1(R)$ in the literature. If R is an A_∞ ring spectrum (over the linear isometry operad), then $H(R)$ is a group-like monoid in \mathcal{M} (see [ABGHR, Section 3.2]). Therefore, one can perform the two-sided bar-construction

$$(5.4) \quad BH(R) = B_{\times_{\mathcal{L}}}(*, H(R)^c, *),$$

where $H(R)^c$ denote the cofibrant replacement of $H(R)$, to obtain the ‘classifying space’ for the principal $H(R)$ -bundle. If R is an E_∞ ring spectrum then $H(R)$ is a grouplike commutative monoid and is represented by a spectrum, call it $h(R)$.

Notation 5.5. The conventional notations for $h(R)$ when $H = \pi_0(R)^\times$ and $H = \{1\}$ are $gl_1(R)$ and $sl_1(R)$, respectively. We will adhere to the conventional notations for the special cases.

Now we explain how to construct Thom spectrum associated to a map in \mathcal{M}

$$f : X \longrightarrow BH(R).$$

Let P be the associated principal $H(R)^c$ -bundle and P' be its cofibrant replacement as a right $H(R)^c$ -module. The spectrum $\mathbb{S}[P']$ admits a right $\mathbb{S}[H(R)^c]$ -module structure. On the other hand there is a natural map

$$\gamma : \mathbb{S}[H(R)^c] \longrightarrow \mathbb{S}[H(R)] \longrightarrow R$$

makes R into a left $\mathbb{S}[H(R)^c]$ -module. The *Thom spectrum* associated to the map $f : X \rightarrow BH(R)$ is the derived smash product

$$Mf = \mathbb{S}[P'] \bigwedge_{\mathbb{S}[H(R)^c]} R$$

(compare [ABGHR, Definition 3.13]). The construction of Thom spectrum is a functor

$$M : \mathcal{M}/_{BH(R)} \longrightarrow \mathcal{S}p_R$$

where $\mathcal{S}p_R$ is the category of R -modules (denoted by \mathcal{M}_R in [ABGHR]).

To distinguish between unstable homotopy groups from stable homotopy groups, we denote the functor that assigns a topological space its n -th unstable homotopy group by

$$\pi_n^u : \mathcal{T}op_* \longrightarrow \text{Groups}.$$

Let \mathcal{M}_* denote the based category of $*$ -modules. The functor π_n^u can be extended to \mathcal{M}_* via the functor F of Equation 5.1. Let \mathcal{S}^n and \mathcal{D}^n denote $L(\mathcal{S}^n)$ and $L(\mathcal{D}^n)$ in \mathcal{M}_* . Then we have

$$\pi_n^u(BH(R)) \cong [\mathcal{S}^n, BH(R)].$$

Notice that $\pi_1^u(BH(R)) = H$. For $n \geq 2$, we have an isomorphism

$$(5.6) \quad \Theta : \pi_n^u(BH(R)) \xrightarrow{\cong} \pi_{n-1}(R).$$

The isomorphism Θ is constructed as follows. By adapting Steenrod's classification theorem in our settings we get,

$$\pi_n^u(BH(R)) \cong \text{Prin}_{H(R)^c}(\mathcal{S}^n),$$

where $\text{Prin}_{H(R)^c}(\mathcal{S}^n)$ denotes the isomorphism classes of principal $H(R)^c$ -bundles. Let \mathcal{D}_+^n and \mathcal{D}_-^n be the northern and the southern hemispheres respectively of the n -sphere

$$\mathcal{S}^n = \mathcal{D}_+^n \cup_{\mathcal{S}^{n-1}} \mathcal{D}_-^n.$$

We also fix a basepoint x_0 of \mathcal{S}^n which is placed on the equator. The principal $H(R)^c$ bundle Pf over \mathcal{S}^n , when restricted to \mathcal{D}_+^n and \mathcal{D}_-^n are trivial bundles as the base spaces are contractible. Thus we get

$$P \cong \mathcal{D}_+^n \times_{\mathcal{L}} H(R)^c \cup_{\theta_f} \mathcal{D}_-^n \times_{\mathcal{L}} H(R)^c$$

where $\theta_f : \mathcal{S}^{n-1} \rightarrow H(R)^c$ is the clutching function defined on the equator, which sends $x_0 \mapsto 1_{H(R)^c}$. In other words, one can think of P as the pushout in the diagram

$$(5.7) \quad \begin{array}{ccc} \mathcal{S}^{n-1} \times_{\mathcal{L}} H(R)^c & \xrightarrow{\tau} & \mathcal{D}_+^n \times_{\mathcal{L}} H(R)^c \\ \downarrow i & & \downarrow \\ \mathcal{D}_-^n \times_{\mathcal{L}} H(R)^c & \xrightarrow{\quad} & P \end{array}$$

where $i(x, g) = (x, g)$ and $\tau(x, g) = (x, \theta_f(x)g)$. Assigning each principal bundle over \mathcal{S}^n a clutching function produces the isomorphism Θ .

Lemma 5.8. *For $\alpha \in \pi_{n-1}(R)$, the Thom spectrum Mf associated to a map $f : \mathcal{S}^n \rightarrow BH(R)$, where $[f] = \Theta^{-1}(\alpha) \in \pi_n^u(BH(R))$ is the cofiber of the map*

$$\alpha \wedge R : \Sigma^{n-1} R \longrightarrow R$$

when $n \geq 2$. For $n = 1$, Mf is the cofiber of the map

$$(1 - \alpha) \wedge R : R \longrightarrow R.$$

Proof. Since $[f] = \Theta^{-1}(\alpha) \in \pi_n(BH(R))$, $[\theta_f]$ must be equal to $\alpha \in \pi_{n-1}^u(H(R)^c)$. Since cofibration is preserved under pushouts and compositions, P in Equation 5.7 is cofibrant. Now we apply the functor $\mathbb{S}[_]_{\mathbb{S}[H(R)^c]}^\wedge R$ to the diagram in Equation 5.7.

When $n \geq 2$, we get Mf as a pushout

$$\begin{array}{ccc} \Sigma^{n-1} R & \xrightarrow{\alpha \wedge R} & R \\ \downarrow 0 & & \downarrow \\ R & \longrightarrow & Mf. \end{array}$$

When $n = 1$, due to the fact that the basepoint maps to the unit component of $H(R)$ we get Mf as the pushout of the diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha \wedge R} & R \\ \downarrow R & & \downarrow \\ R & \longrightarrow & Mf. \end{array}$$

The result follows from the above diagrams. \square

Let $\mathcal{S}_{\mathbb{S}_p}^0$ be the cofibrant replacement for the p -adic sphere spectrum \mathbb{S}_p^0 . The units in $\pi_0(\mathcal{S}_{\mathbb{S}_p}^0) \cong \hat{\mathbb{Z}}_p$, the p -adic integers, are

$$(\hat{\mathbb{Z}}_p)^\times \cong \begin{cases} \mathbb{Z}/(p-1) \times \hat{\mathbb{Z}}_p & \text{if } p \neq 2 \\ \mathbb{Z}/2 \times \hat{\mathbb{Z}}_2 & \text{if } p = 2. \end{cases}$$

Let G_p denote the subgroup $\{1\} \times \hat{\mathbb{Z}}_p < (\hat{\mathbb{Z}}_p)^\times$. When p is an odd prime, the elements of G_p as a subset of $\hat{\mathbb{Z}}_p$ are precisely those which are of the form $1 + p\hat{\mathbb{Z}}_p$. When $p = 2$, G_2 consists of elements of the form $1 + 4\hat{\mathbb{Z}}_2$. For convenience, we use the following notations.

Notation 5.9. We will denote the space $G_p(\mathcal{S}_{\mathbb{S}_p}^0)^c$, the cofibrant replacement $G_p(\mathcal{S}_{\mathbb{S}_p}^0)$, by \mathcal{G}_p and the corresponding spectrum $g_p(\mathcal{S}_{\mathbb{S}_p}^0)$ by g_p .

Note that $\pi_1^u(B\mathcal{G}_p) = G_p$. The following corollary is an easy consequence of Lemma 5.8.

Corollary 5.10. *The Thom spectrum associated to the map $f : \mathcal{S}^1 \rightarrow B\mathcal{G}_p$ which represents the homotopy class of $1 + p^i e \in \pi_1^u(B\mathcal{G}_p)$, where $e \in \hat{\mathbb{Z}}_p^\times$, is the Moore spectrum $M_p(i)$.*

One can detect the A_n -structure on a Thom spectrum using the work of Lewis in [LMS, §IX]. Let \mathcal{O} be an operad. Lewis worked in Lewis-May-Steinberger [LMS] category of spectra and showed that the Thom spectrum associated to an \mathcal{O} -map

$$f : X \longrightarrow BF$$

where $F \simeq SL_1(\mathbb{S}^0)$, admits an \mathcal{O} -structure. In [ABGHR], the authors proved a version of Lewis' result in the [EKMM] category of S -modules, where they replace F with general grouplike objects $GL_1(R)$ or $SL_1(R)$ for E_∞ ring spectrum R , but restrict themselves to E_∞ -structures only. However, combining the work of Lewis in [LMS, §IX] and [ABGHR] one can obtain the following result:

Theorem 5.11 (Lewis). *Let X be an H -space which admits a unital A_n -structure and admits a unital A_n -map*

$$f : X \longrightarrow B\mathcal{G}_p$$

then the Thom spectrum Mf inherits a unital A_n -structure.

Here is a brief nontechnical explanation of the proof of Theorem 5.11. The fact that f is a unital A_n -map means that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{K}^i \times X^{\times i} & \xrightarrow{\mathcal{K}^i \times f^{\times i}} & \mathcal{K}^i \times B\mathcal{G}_p^{\times i} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B\mathcal{G}_p \end{array}$$

for $0 \leq i \leq n$. Since the Thom spectrum associated to the map $\mu_n(\mathcal{K}^i \times f^{\times i})$ is $\mathbb{S}[\mathcal{K}^i] \wedge (Mf)^{\wedge i}$, the above diagram yields maps

$$\mu_i : \mathbb{S}[\mathcal{K}^i] \wedge (Mf)^{\wedge i} \longrightarrow Mf$$

for $0 \leq i \leq n$. It can be shown that the maps μ_i for $0 \leq i \leq n$ induce a unital A_n -structure on Mf .

Remark 5.12 (Two different bar constructions). The obstruction theory for A_n -structures in $\mathcal{T}op$ via Stasheff's bar construction as discussed in Section 4.4, applies to the category of $*$ -modules \mathcal{M} via the functor L . As a result, we now have two different bar complex for objects with A_∞ -structure in \mathcal{M} , one coming from Equation 5.4 and the other from Stasheff's construction in Definition 4.24. However, for cofibrant objects in \mathcal{M} with A_∞ -structure, two different bar constructions yield isomorphic objects in the homotopy category $h\mathcal{M}$. Therefore, for the expediency of notations, we will not distinguish the two different bar complexes.

Proof of Main Theorem 1. By Corollary 5.10, the Moore spectrum $M_p(i)$ is the Thom spectrum associated to any map

$$f_p(i) : \mathcal{S}^1 \longrightarrow B\mathcal{G}_p,$$

which belongs to the homotopy class $1 + p^i e \in \pi_1^u(B\mathcal{G}_p)$ for some $e \in \hat{\mathbb{Z}}_p^\times$. To obtain a unital A_n -structure on $M_p(i)$, it suffices to find a map $f_p(i)$ in the homotopy class of $1 + p^i e$ which is a unital homotopy A_n -map. Indeed, by Theorem 4.33, we can always replace $f_p(i)$ by a unital A_n -map, therefore by Theorem 5.11, $M_p(i)$ gets a unital A_n -structure.

By Theorem 4.31, $f_p(i)$ is a unital homotopy A_n -map if and only if $\Sigma f_p(i)$ extends to a map

$$\begin{array}{ccc} \mathcal{S}^2 & \xrightarrow{\Sigma f_p(i)} & \Sigma B\mathcal{G}_p \\ \downarrow t & & \downarrow \\ B_n \mathcal{S}^1 \simeq \mathbb{C}P^n & \longrightarrow & B_n B\mathcal{G}_p \end{array}$$

or equivalently, the composite $\mathcal{S}^2 \xrightarrow{\Sigma f_p(i)} \Sigma B\mathcal{G}_p \rightarrow B^2\mathcal{G}_p$, name it $\tilde{f}_p(i)$, factors through $\mathbb{C}P^n$

$$(5.13) \quad \begin{array}{ccc} \mathcal{S}^2 & \xrightarrow{\tilde{f}_p(i)} & B^2\mathcal{G}_p \\ \downarrow t & \nearrow f_p^n(i) & \\ \mathbb{C}P^n & & \end{array}$$

Note that, it is enough to solve this lifting problem in the homotopy category of \mathcal{M}_* , i.e. the homotopy class $[\tilde{f}_p(i)] = 1 + p^i e \in \pi_2^u(B^2\mathcal{G}_p)$ lifts to a class $x \in [\mathbb{C}P^n, B^2\mathcal{G}_p]$. If the diagram in Equation 5.13 commutes up to homotopy, then we can always arrange $\tilde{f}_p(i)$ and $f_p^n(i)$ in the homotopy class of $1 + p^i e$ and x respectively such that the diagram commutes in \mathcal{M}_* . This can be done by choosing the map $t : \mathcal{S}^2 \rightarrow \mathbb{C}P^n$ to be a cofibration and using the homotopy extension property. Since $\mathcal{G}_p = \Omega^\infty g_p$ is an infinite loop space, the above problem reduces to the lifting problem

$$\begin{array}{ccc} \mathbb{S}^0 & \xrightarrow{1+p^i e} & g_p, \\ \downarrow & \nearrow & \\ \Sigma^{\infty-2}\mathbb{C}P^n & & \end{array}$$

in the homotopy category of $\mathcal{S}p$. Therefore, to find unital A_n -structures on $M_p(i)$, we try to solve the above lifting problem.

Note that the composite

$$\mathcal{S}^1 \xrightarrow{m} \mathcal{S}^1 \xrightarrow{1+p} B\mathcal{G}_p$$

is the map $(1+p)^m$. Therefore the action of \mathbb{Z} (i.e. $\pi_0(\mathbb{S}^0)$) on the generator $1+p \in \pi_0(g_p) \cong \hat{\mathbb{Z}}_p$ is given by the formula

$$m \cdot (1+p) = (1+p)^m.$$

It is well-known that for an odd prime p ,

$$(1+p)^{p^i} = 1 + p^{i+1}e$$

and

$$(1+2)^{2^i} = 1 + 2^{i+2}e$$

for some $e \in \hat{\mathbb{Z}}_p^\times$. Therefore we fix n and indulge ourselves in finding an estimate on i for which $p^i \cdot (1+p) \in \pi_0(\mathcal{G}_p)$ is in the image of the map

$$t^* : [\Sigma^{\infty-2}\mathbb{C}P^n, g_p] \longrightarrow [\mathbb{S}^0, g_p].$$

The element $p^i \cdot (1+p)$ is in the image t^* if and only if $p^i \cdot (1+p) \in E_2^{0,0}$ survives the Atiyah Hirzebruch spectral sequence

$$(5.14) \quad E_2^{l,r} : H^l(\Sigma^{\infty-2}\mathbb{C}P^n; \pi_r(g_p)) \Rightarrow [\Sigma^{\infty-2}\mathbb{C}P^n, g_p]_{l-r}.$$

The relevant part of the E_2 -page of this spectral sequence is drawn below. There are only finitely many r for which the differentials

$$d_{2r} : E_2^{0,0} \cong \hat{\mathbb{Z}}_p \longrightarrow E_2^{2r,-r+1} \cong \pi_{2r-1}(g_p),$$

as $E_2^{l,r} = 0$ for $l > 2n-2$. Moreover, each differential may kill at most finitely many powers of p of the generator $1+p$, as $\pi_k(g_p) \cong \pi_k(\mathbb{S}_p^0)$ has finite p -torsions for

| | 0 | 1 | 2 | 3 | ... | $2n-2$ | $2n-1 \dots$ |
|-----------|----------------------|----------|----------------------|----------|-----|----------------------|--------------|
| 0 | $\hat{\mathbb{Z}}_p$ | 0 | $\hat{\mathbb{Z}}_p$ | 0 | ... | $\hat{\mathbb{Z}}_p$ | 0 ... |
| -1 | $\pi_1(g_p)$ | 0 | $\pi_1(g_p)$ | 0 | ... | $\pi_1(g_p)$ | 0 ... |
| -2 | $\pi_2(g_p)$ | 0 | $\pi_2(g_p)$ | 0 | ... | $\pi_2(g_p)$ | 0 ... |
| -3 | $\pi_3(g_p)$ | 0 | $\pi_3(g_p)$ | 0 | ... | $\pi_3(g_p)$ | 0 ... |
| \vdots | \vdots | \vdots | \vdots | \vdots | ... | \vdots | \vdots |
| $-(2n-3)$ | $\pi_{2n-3}(g_p)$ | 0 | $\pi_{2n-3}(g_p)$ | 0 | ... | $\pi_{2n-3}(g_p)$ | 0 ... |

$k > 0$. Though we do not have any knowledge of the differentials, one can sharpen the bound through the following claim.

Claim 2. For any $x \in E_2^{0,0}$ we have $d_{2r}(px) = pd_{2r}(x) = 0 \in E_2^{2r,2r-1}$.

Firstly the differentials of the Atiyah Herzebruch spectral sequence are \mathbb{Z} linear therefore

$$d_{2r}(px) = pd_{2r}(x).$$

Also, the Atiyah Herzebruch spectral sequence

$$E_2^{l,r} = H^l(X; \pi_r(Y)) \Rightarrow [X, Y]_{l-r}$$

is natural in the variable X . Since, the multiplication by p map on $\Sigma^{\infty-2}\mathbb{C}P^n$ induces multiplication by p^{r+1} on $H^r(\Sigma^{\infty-2}\mathbb{C}P^n)$, we have

$$d_{2r}(px) = p^{r+1}d_{2r}(x).$$

Combining the two equations, we get

$$p(1 - p^r)d_{2r}(x) = 0.$$

Since $1 - p^r$ is a unit in $\hat{\mathbb{Z}}_p$, we have $pd_{2r}(x) = 0$. This completes the proof of Claim 2.

Consequently, at most one power of p on the generator $(1 + p)$ is killed if $\pi_{2r-3}(S_p^0) \neq 0$. Thus, $p^{o_p(n)+1} \cdot (1 + p)$ survives the spectral sequence, where

$$o_p(n) = \#\{k \leq 2n-3 \text{ and odd} : \pi_k(S_p^0) \neq 0\}$$

and the result follows. \square

Remark 5.15. One should note that the complete knowledge of the differentials in the spectral sequence 5.14 will give the A_n -structure that the Moore spectra inherits by virtue of being a Thom spectrum. The upper bound on n for which $M_p(i)$ supports A_n -structure may not be obtained from its Thom spectrum structure. Author believes that $M_2(2)$ and $M_2(3)$ can turn out to be examples of such a situation. It will be interesting to know if such examples exist at an odd prime.

We will end the section discussing the following conjecture.

Conjecture 2. For an odd prime p , the obstruction to the A_n -structure on $M_p(i)$ lies in the image of J part, i.e. the chromatic layer 1, in the stable homotopy groups of $M_p(i)$.

Let p be an odd prime. Let \mathcal{J} denote the connected cover of the image of the J -homomorphism spectrum. It is known that there is a map

$$\psi : \mathcal{J} \longrightarrow sl_1(\mathbb{S}^0).$$

After p -completion it may be possible to extend the map ψ to a map

$$\psi_p : \mathcal{J}_p \longrightarrow g_p$$

for such appropriate \mathcal{J}_p such that $\pi_0(\psi_p)$ is an isomorphism. In particular, the spectrum \mathcal{J}_p should have the property that $\pi_0(\mathcal{J}_p) \cong \hat{\mathbb{Z}}_p$ and $\Omega^\infty \mathcal{J}$ is included in $\Omega^\infty \mathcal{J}_p$ as the zero component of \mathcal{J}_p . An ideal candidate for \mathcal{J}_p is $h(L_{K(1)}\mathbb{S}_p^0)$, the delooping of $H(L_{K(1)}\mathbb{S}_p^0)$, where $H = \{1\} \times \hat{\mathbb{Z}}_p \subset \hat{\mathbb{Z}}_p^\times$. The existence of the map ψ_p is currently under investigation by the author and N. Kitchloo.

However, if ψ_p exists, then we will have the factorization

$$f_p(i) : \mathcal{S}^1 \longrightarrow \Omega^{\infty-1} \mathcal{J}_p \longrightarrow B\mathcal{G}_p.$$

Consequently, the obstruction to A_n -structure of $M_p(i)$ will be in the image of J part of the stable homotopy groups of spheres. An evidence for such a phenomena is that of $M_p(1)$, whose obstruction to A_p -structure is $\alpha_1 \in \pi_{2p-3}(M_p(1))$, which is in the image of J .

Just the existence of \mathcal{J}_p does not completely prove Conjecture 2: There is another caveat to this problem. The existence of \mathcal{J}_p will show that the obstruction to A_n -structures on $M_p(i)$ which arise from its Thom spectrum structure lies in the image of J part of $\pi_*(M_p(i))$. As pointed out earlier, not all A_n -structures may arise this way and we expect $M_2(2)$ and $M_2(3)$ to be examples of such a phenomenon. Therefore the case when $p = 2$ is ruled out from Conjecture 2. If the conjecture is true, it will raise the question whether this phenomena propagates to the higher chromatic layers, i.e. the obstruction to higher associativities of a type n spectrum are associated to elements of $\pi_*(\mathbb{S}^0)$ in the chromatic layer n or above.

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